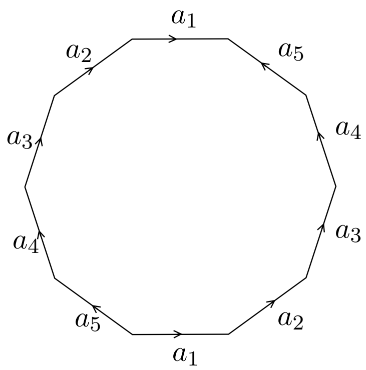



Figure 2.2 Rotation by $2\pi/10$ gives an order 10 element of $\mathrm{Mod}(S_2)$.

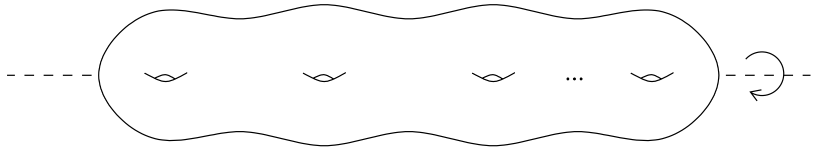



Figure 2.3 The rotation by $\pi$ about the indicated axis is a hyperelliptic involution.

example, what does an order 5 symmetry of $S_2$ look like with respect to the standard picture of $S_2$ embedded in $\mathbb{R}^3$?

Unlike the preceding examples, most elements of the mapping class group have infinite order. The simplest such elements are Dehn twists, which are defined and studied in detail in Chapter 3.

## 2.2 COMPUTATIONS OF THE SIMPLEST MAPPING CLASS GROUPS

In this section we give complete descriptions of the mapping class groups of the simplest surfaces, working directly from the definitions.

### 2.2.1 THE ALEXANDER LEMMA

Our first computation is the mapping class group $\mathrm{Mod}(D^2)$ of the closed disk $D^2$. This simple result underlies most computations of mapping class groups.

**Lemma 2.1 (Alexander lemma)** *The group* $\mathrm{Mod}(D^2)$ *is trivial.*

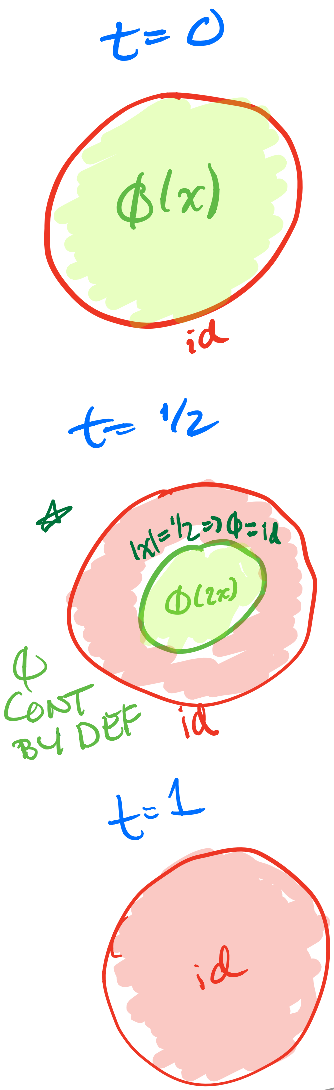
In other words, Lemma 2.1 states that given any homeomorphism $\phi$ of $D^2$ that is the identity on the boundary $\partial D^2$, there is an isotopy of $\phi$ to the identity through homeomorphisms that are the identity on $\partial D^2$.

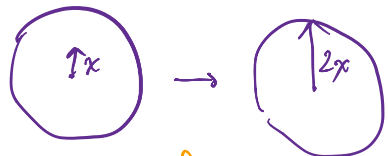
*Proof.* Identify $D^2$ with the closed unit disk in $\mathbb{R}^2$. Let $\phi : D^2 \to D^2$ be a homeomorphism with $\phi|_{\partial D^2}$ equal to the identity. We define

$$F(x,t) = \begin{cases} (1-t)\phi\left(\frac{x}{1-t}\right) & 0 \le |x| < 1-t \\ x & 1-t \le |x| \le 1 \end{cases}$$

for $0 \le t < 1$, and we define $F(x,1)$ to be the identity map of $D^2$. The result is an isotopy $F$ from $\phi$ to the identity. $\square$

⇒ isotoped $\phi$ into id. USING DILATION PROPERTY of circle




We can think of combining the $\{F(x,t)\}$ from the proof into a level-preserving homeomorphism of a cylinder whose support is a cone; see Figure 2.4. The individual $F(\star, t)$ homeomorphisms appear at horizontal slices.

Green = Supp$(F(x_0,t))$

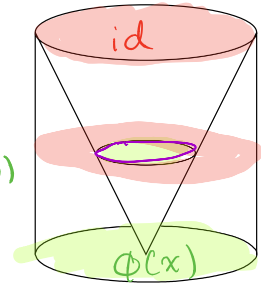



Figure 2.4 The Alexander trick.

hyp: any surface S homeomorphic to Disk has Mod(S) = 1 ?

The isotopy given by the proof can be thought of as follows: at time $t$, do the original map $\phi$ on the disk of radius $1-t$ and apply the identity map outside this disk. This clever proof is called the Alexander trick.

The reader will notice that the Alexander trick works in all dimensions. However, this is one place where it is convenient to think about homeomorphisms instead of diffeomorphisms. The smooth version of the Alexander lemma in dimension 2 is not nearly as simple, although in this case Smale proved the stronger statement that $\mathrm{Diff}(D^2, \partial D^2)$ is contractible [197]. In higher dimensions, the situation is worse: it is not known if $\mathrm{Diff}(D^4, \partial D^4)$ is connected, and for infinitely many $n$ we have that $\mathrm{Diff}(D^n, \partial D^n)$ is not connected.

N-d DISKS

TELLING US THAT IT'S UNKNOWN if use diffeo instead of homeo... so cannot ascertain MCG(S) for S is dif'able?

The proof of Lemma 2.1 also holds with $D^2$ replaced by a once-punctured disk (take the puncture/marked point to lie at the origin), and hence we also have the following:

*The mapping class group of a once-punctured disk is trivial.*

g = genus
n = # of punctures

1-PUNCT sphere


• = ∞


**The sphere and the once-punctured sphere.** There are two other mapping class groups $\mathrm{Mod}(S_{g,n})$ that are trivial, namely, $\mathrm{Mod}(S_{0,1})$ and $\mathrm{Mod}(S^2)$. For the former, we can identify $S_{0,1}$ with $\mathbb{R}^2$ and use that fact that every orientation-preserving homeomorphism of $\mathbb{R}^2$ is homotopic to the identity via the straight-line homotopy. For $S^2$, any homeomorphism can be modified by isotopy so that it fixes a point, and so we can apply the previous example.

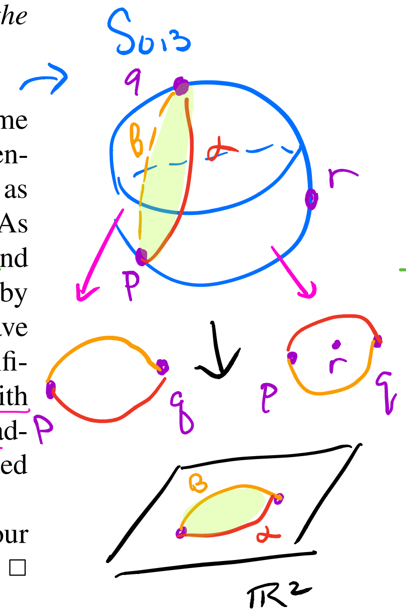
### 2.2.2 The Mapping Class Group of the Thrice-punctured Sphere

Our next example, the mapping class group of $S_{0,3}$, illustrates an important idea in the theory of mapping class groups. The way we will compute $\mathrm{Mod}(S_{0,3})$ is to understand its action on some fixed arc in $S_{0,3}$. The surface obtained by cutting $S_{0,3}$ along this arc is a punctured disk, and so we will be able to apply the Alexander lemma. This is in general how we use the cutting procedure for surfaces in order to perform inductive arguments.

SAYING GOING TO CUT SURFACES INTO $S_{0,3}$'s.

In this section it will be convenient to think of $S_{0,3}$ as a sphere with three marked points (instead of three punctures). In order to determine $\mathrm{Mod}(S_{0,3})$ we first need to understand simple proper arcs in $S_{0,3}$.

**Proposition 2.2** *Any two essential simple proper arcs in $S_{0,3}$ with the same endpoints are isotopic. Any two essential arcs that both start and end at the same marked point of $S_{0,3}$ are isotopic.*




*Proof.* Let $\alpha$ and $\beta$ be two simple proper arcs in $S_{0,3}$ connecting the same two distinct marked points. We can modify $\alpha$ by isotopy so that it has general position intersections with $\beta$. By thinking of the third marked point as being the point at infinity, we can think of $\alpha$ and $\beta$ as arcs in the plane. As in the proof of Lemma 1.8, if $\alpha$ and $\beta$ are not disjoint, then we can find an innermost disk bounded by an arc of $\alpha$ and an arc of $\beta$. Pushing $\alpha$ by isotopy across such disks, we may reduce intersection until $\alpha$ and $\beta$ have disjoint interiors. At this point, we can cut $S_{0,3}$ along $\alpha \cup \beta$. By the classification of surfaces, the resulting surface is the disjoint union of a disk (with two marked points on the boundary) and a once-marked disk (with two additional marked points on the boundary). Thus $\alpha$ and $\beta$ bound an embedded disk in $S_{0,3}$, and so they are isotopic.

most of this is resolving technicalities for α, β.

The case where $\alpha$ and $\beta$ are essential simple proper arcs where all four endpoints lie on the same marked point of $S_{0,3}$ is similar. □

We are now ready to compute $\mathrm{Mod}(S_{0,3})$. Let $\Sigma_3$ denote the group of permutations of three elements.

$S_3$

PROOF OF KERNEL = 1:

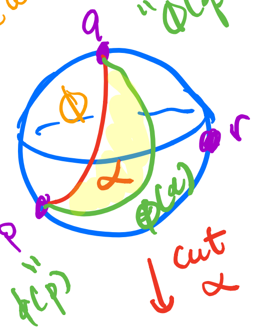
**Proposition 2.3** *The natural map*

$$\mathrm{Mod}(S_{0,3}) \to \Sigma_3$$

*given by the action of* $\mathrm{Mod}(S_{0,3})$ *on the set of marked points of* $S_{0,3}$ *is an isomorphism.*

BECAUSE HAVE 3 MARKINGS?

NEW $\phi$ fixes $\alpha$




cut $\alpha$

*Proof.* The map in the statement is obviously a surjective homomorphism. Thus it suffices to show that if a homeomorphism $\phi$ of $S_{0,3}$ fixes the three marked points—call them $p$, $q$, and $r$—then $\phi$ is homotopic to the identity. Choose an arc $\alpha$ in $S_{0,3}$ with distinct endpoints, say $p$ and $q$. Since $\phi$ fixes the marked points $p$, $q$, and $r$, the endpoints of $\phi(\alpha)$ are again $p$ and $q$. By Proposition 2.2, we have that $\phi(\alpha)$ is isotopic to $\alpha$. It follows that $\phi$ is isotopic to a map (which we also call $\phi$) that fixes $\alpha$ pointwise (Proposition 1.11).

KERNEL = {1}

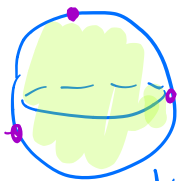
We can cut $S_{0,3}$ along $\alpha$ so as to obtain a disk with one marked point (the boundary comes from $\alpha$, and the marked point comes from $r$). Since $\phi$ preserves the orientations of $S_{0,3}$ and of $\alpha$, it follows that $\phi$ induces a homeomorphism $\overline{\phi}$ of this disk which is the identity on the boundary (the map $\overline{\phi}$ is the unique set map on the cut-open surface inducing $\phi$). By Lemma 2.1, the mapping class group of a once-marked disk is trivial, and so $\overline{\phi}$ is homotopic to the identity. The homotopy induces a homotopy from $\phi$ to the identity. □


$\overline{\phi}$ $\mathrm{Mod}(D^2)=1$

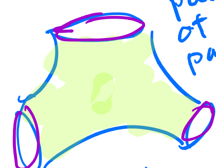
"cut" just psychologically describes $\phi$ action.

$F: \phi \to id$ ✓.

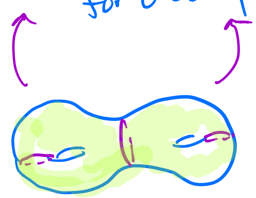
$S_{0,3}$



homeo pair of pants



(fundamental object for decomp)



**Pairs of pants.** The surface $S_{0,3}$ is homeomorphic to the interior of a *pair of pants*[1] $P$, which is the compact surface obtained from $S^2$ by removing three open disks with embedded, disjoint closures. Pairs of pants are important because all compact hyperbolic surfaces can be built from pairs of pants (cf. Section 10.5). In Section 3.6, we will apply Proposition 2.3 to show that $\mathrm{Mod}(P) \approx \mathbb{Z}^3$.

**The twice-punctured sphere.** There is a homomorphism $\mathrm{Mod}(S_{0,2}) \to \mathbb{Z}/2\mathbb{Z}$ given by the action on the two marked points. An analogous proof to that of Proposition 2.3 gives that $\mathrm{Mod}(S_{0,2}) \approx \mathbb{Z}/2\mathbb{Z}$.

### 2.2.3 The Mapping Class Group of the Annulus

We now come to the simplest infinite-order mapping class group, that of the annulus $A$. The basic procedure we use to compute $\mathrm{Mod}(A)$ is similar to the one we used for $S_{0,3}$. That is, we find an arc in $A$ so that when we cut

[1]Möbius used the term "trinion" for a pair of pants (he called an annulus a "binion" and a disk a "union").