Formal Languages in Left-Orderable Groups

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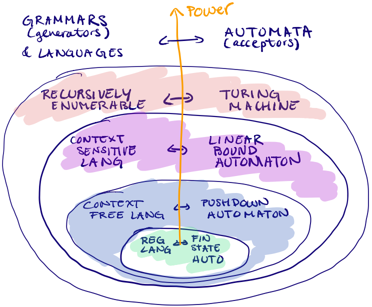
Goals

Goal of PhD: To classify *positive cones* of groups in terms of the minimal *formal language* complexity representation with respect to the Chomsky hierarchy.



You don't have to know what any of this means :)!

Goal of Talk: I will give a flavor.

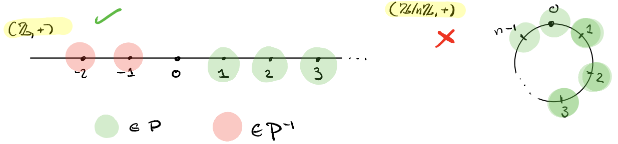
Basic Definitions

Roughly speaking, a positive cone is trying to capture the notion of additive positivity.

$P \subset G$ is a *positive cone* for G if

- $PP \subseteq P$ (closed under semigroup operation),
- $G = P \sqcup P^{-1} \sqcup \{1_G\}$ (trichotomy property).

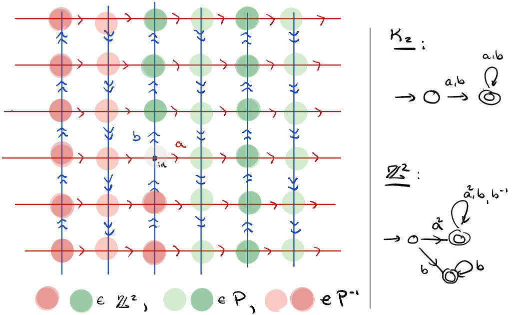
Ex: $\mathbb{Z}$ admits *a* positive cone (ex given below), whereas $\mathbb{Z}/n\mathbb{Z}$ does not.



For those who are familiar with left orders:
admitting a left-order and admitting a positive cone are equivalent.

Toy Example: K_2 and $\mathbb{Z}^2$

$K_2 = \langle a, b : a^{-1}bab = 1\rangle$ has index two subgroup $\mathbb{Z}^2 = \langle a^2, b\rangle$.



- K_2 and $\mathbb{Z}^2$ admit positive cones described by the language accepted by the automata on the right. We call such positive cones *regular*.
- K_2's positive cone language is actually finitely generated (which is stronger than regular) and can be written $\langle a, b\rangle^+$, whereas the inherited positive cone for $\mathbb{Z}^2$ is actually not finitely generated.

Result 1: Passing to Finite Index Subgroups

Takeaway: Being finitely generated as a positive cone is not passed down to finite index subgroups!

Theorem 1 [Su19]

Regular positive cones pass to finite index subgroups.

By the toy example with $\mathbb{Z}^2 \leq K_2$, this is the minimal complexity that is passed down to finite index.

Next: Let's get some intuition for why $\mathbb{Z}^2$ does not admit a finitely generated positive cone.

Topology of Space of Positive Cones

Take $\mathrm{LO}(G)$ to be the spaces of positive cones of G.

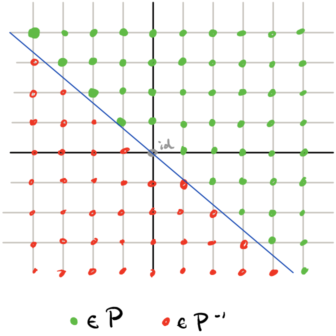
- All $P \in \mathrm{LO}(G)$ are subsets of $G - \{1_G\}$,
  $\implies \mathrm{LO}(G)$ is a subset of the power set of $G - \{1_G\}$.
- $\mathrm{LO}(G)$ inherit the natural product topology on $\subseteq \{+1, -1\}^{G-\{1_G\}}$.

Point: Two positives cones are close in that space if they agree on large balls of group elements based at 1_G.

P is finitely generated $\implies$ P is an isolated point.

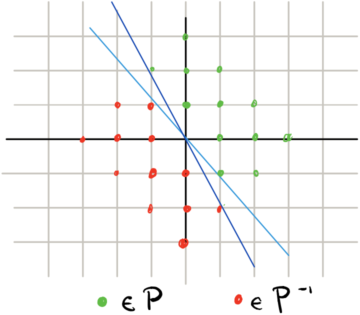
Positive cones of $\mathbb{Z}^2$

The positive cones in $\mathbb{Z}^2$ are defined by the half-spaces and a choice of orientations. This half-space represents a point in $\mathrm{LO}(\mathbb{Z}^2)$.



Neighbourhoods of $\mathrm{LO}(\mathbb{Z}^2)$

A *neighbourhood* in $\mathrm{LO}(\mathbb{Z}^2)$ is the set of all positive cones agreeing on a finite ball, but which do not necessarily coincide outside of the ball.



$\implies$ $\mathrm{LO}(\mathbb{Z}^2)$ has no isolated points.

$\implies$ $\mathbb{Z}^2$ has no finitely generated positive cones! $\square$

(because P is finitely generated $\implies$ P is an isolated point.)

Result 2: F_2 vs $F_2 \times \mathbb{Z}$

Counterintuitive example:

- F_2 does not admit a regular positive cone [HŠ17].
- $B_3 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$, admits a regular positive cone [Nav11].
- However, $F_2 \times \mathbb{Z}$ is a finite index subgroup of B_3.
- $\implies F_2 \times \mathbb{Z}$ has a regular positive cone (by Theorem 1).

It was known that $\mathrm{LO}(F_2 \times \mathbb{Z})$ had an isolated point [MR18].

Theorem 2 [Su19]

In fact, $F_2 \times \mathbb{Z}$ is part of a new infinite family of groups with finitely generated positive cone; where for $m \geq 3$, there is an infinite number of groups with positive cone of rank m [Su19].

This answers a question of Navas [Nav11].

Result 3: Positive Cones of Acylindrically Hyperbolic Groups

Theorem 3 [Su19]

A quasi-geodesic positive cone language of a finitely generated acylindrically hyperbolic group cannot be regular.

Short Bibliography

[HŠ17]: Hermiller and Šunić. DOI: 10.1142/S0218196717500527
[MR18]: Mann and Rivas. DOI: 10.5802/aif.3191
[Nav11]: Navas. DOI: 10.1016/j.jalgebra.2010.10.020
[Su19]: Su. arXiv: 1905.13001