Formal language convexity in Left-Orderable Groups

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Overarching Goal

Classifying positive cones of groups in terms of the minimal formal language complexity (w.r.t Chomsky hierarchy) required to represent them.

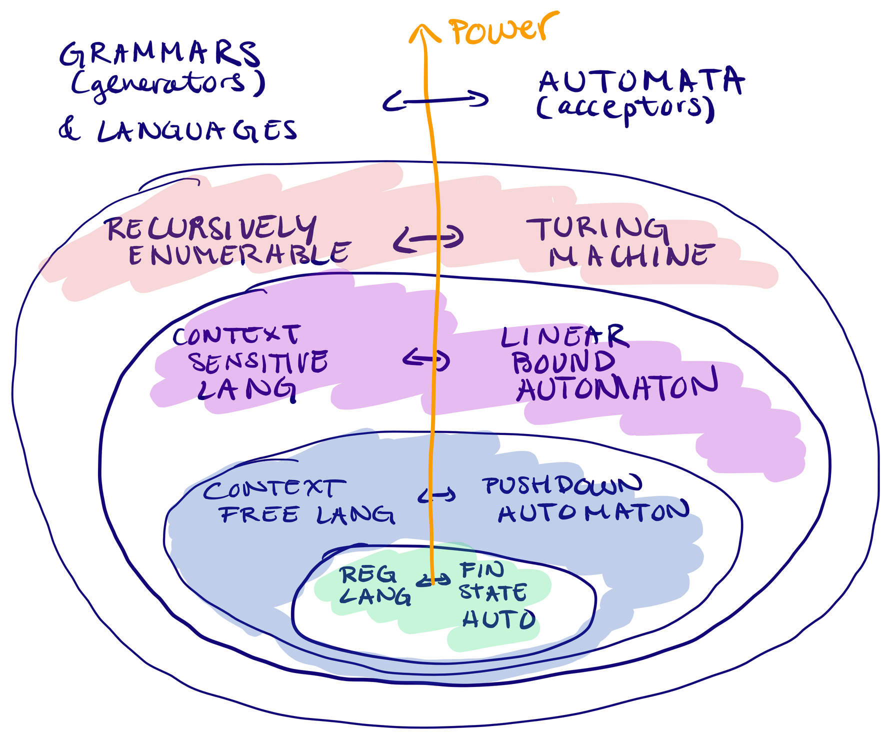


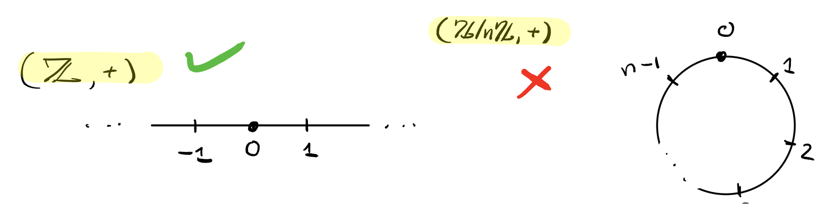
Fig: Chomsky hierarchy.

Basic Definitions

- $P \subset G$ is a *positive cone* for G if
  1. $PP \subseteq P$ (closed under semigroup operation),
  2. $G = P \sqcup P^{-1} \sqcup \{1\}$ (trichotomy property).
- A group G is *left-orderable* if there exists a total order on the group that is invariant by left-multiplication, so $\forall g, h, f \in G$,

$$g < h \iff fg < fh.$$

Ex: $\mathbb{Z}$ is orderable; $\mathbb{Z}/n\mathbb{Z}$ is not.



- Being left-orderable and admitting a positive cone are equivalent.
  - P defines a left-invariant order by

$$g < h \iff g^{-1}h \in P.$$

  - A left-invariant order $<$ defines a positive cone by

$$P := \{g \in G : g > 1\}.$$

- Let G be finitely generated (f.g.) by X. A *language* is a subset in the monoid X^*. A language is *regular* if it is accepted by a finite state automaton. A *positive cone language* (PCL) for G is a language which evaluates in G to a positive cone.
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Positive cone language examples: K_2 and $\mathbb{Z}^2$

$K_2 = \langle a, b : a^{-1}bab = 1\rangle$ has index two subgroup $\mathbb{Z}^2 = \langle a^2, b\rangle$.

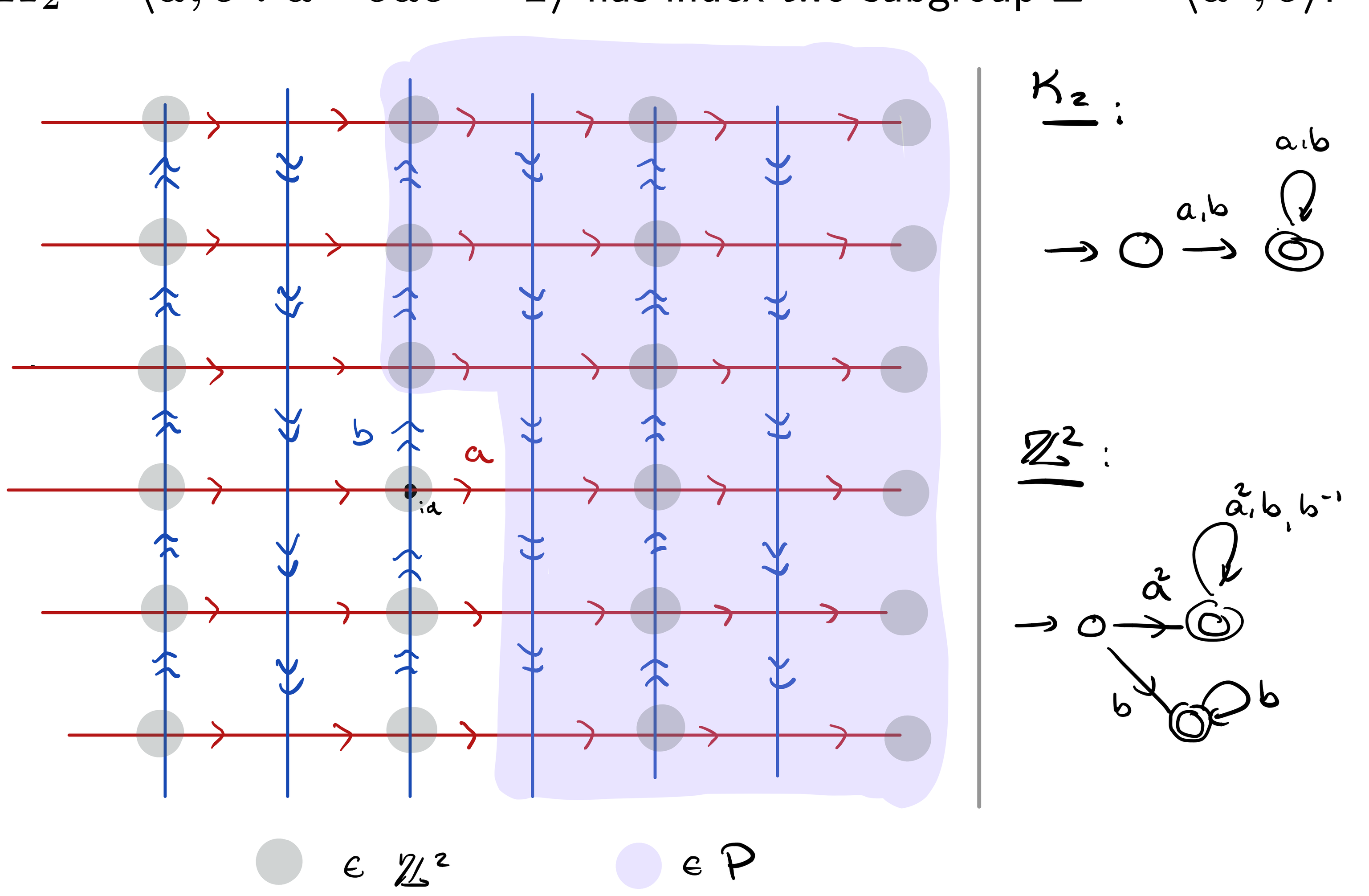


Fig: Left: Cayley graph and positive cone for K_2; right: finite state automata for associated PCLs of K_2 and $\mathbb{Z}^2$.

- K_2 admits regular PCL $\langle a, b\rangle^+$.
- $\mathbb{Z}^2$ admits regular PCL $bb^* \cup a^2(a^2 \cup b \cup b^{-1})^*$.

Remark: It can be shown $\mathbb{Z}^2$ cannot have a f.g. positive cone, as its space of left-orders is homeomorphic to the Cantor set. Thus, both PCLs are optimal in complexity.

Result 1: Groups with Finitely Generated Positive Cones

Context: there are not many known examples of groups admitting f.g. positive cones. Navas [Nav11] showed that there is an infinite family of 2-generated positive cones, $\Gamma_n = \langle a, b \mid ba^n b = a\rangle$ with positive cones $P_n = \langle a, b\rangle^+$.
Note: $\Gamma_1 = K_2$.

Navas' Question : for every $k \geq 3$, find an infinite family of groups which admit a positive cone of rank k (k-generated).

- **Theorem** [Su19] **:** For every integer $m \geq 2$, and integer $n \geq 2$ of the form $n = m - 1 + mt$ for some odd integer t, there is a subgroup of index m in Γ_n which admits a positive cone of rank $m + 1$.

Proof Sketch: Let H be given by $\ker \varphi$, where φ is a homomorphism $\Gamma_n \to \mathbb{Z}/m\mathbb{Z}$ such that $\varphi(a) = \varphi(b) = 1$. This is possible by choice of m and n. Use the Reidemeister-Schreier method on H with transversal $T = \{b^0, b^{-1}, \ldots, b^{-(m-1)}\}$. Observe that the image of P_n is anti-symmetric, showing that H has a f.g. positive cone. Abelianize H and observe that the rank of Ab H is $m + 1$.

Result 2: Positive Cones of Finite Index Subgroup

- **Theorem** [Su19] **:** Let G be a f.g. group which admits a regular PCL. If H is a finite index (f.i.) subgroup, then H admits a regular PCL.

Ex: $B_3 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle$ has $F_2 \times \mathbb{Z}$ as a f.i. subgroup.
Note: F_2 does not admit a regular PCL [HŠ17].

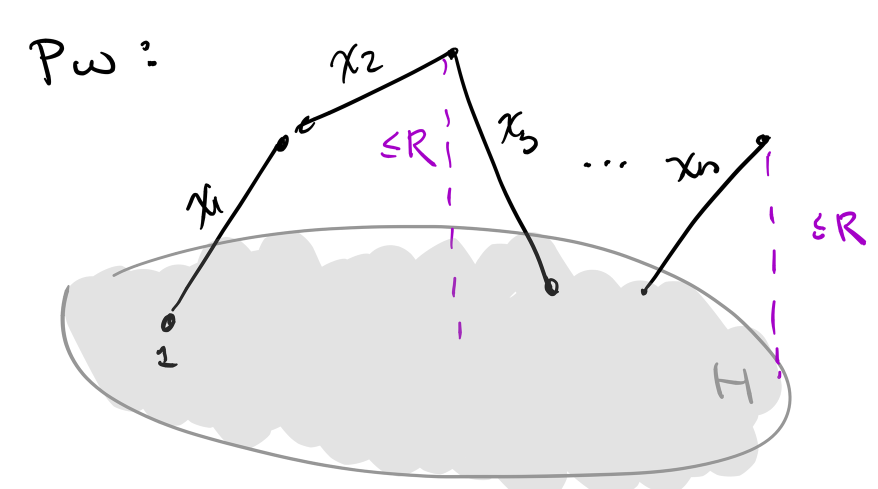
Observation: Regularity is the lowest complexity which can be passed to finite index subgroups: K_2 example shows that finite generation is not passed down.

Proof Sketch: Finite index subgroups are L-convex.

L-convexity

- **Definition** [Su19] **:** Let L be a language over X. A subset $H \subseteq G$ is *L-convex* if there exists an $R \geq 0$ such that for all $w \in L$, the induced path p_w lies within distance R of H in Γ.

$w \in L$, $w = x_1 x_2 \ldots x_n$



- **Proposition** [Su19] Let L be a regular language, and let $P = \pi(L)$ where π is the evaluation map onto a f.g. group G. Let H be a subgroup of G. If H is L-convex, then there exists a regular language L_H such that $\pi(L_H) = H \cap P$.

Result 3: Positive Cones of Acylindrically Hyperbolic Groups

- **Theorem** [Su19] **:** A quasi-geodesic PCL of a f.g. acylindrical hyperbolic group cannot be regular.

Proof Sketch: Let G be an acylindrically hyperbolic group. Then there exists a hyperbolically embedded subgroup H of G that is isomorphic to F_2. Since H is hyperbolically embedded, H has the Morse property. Assume for contradiction that G admits quasi-geodesic regular PCL L. By Morse property, H is L-convex. This means that there exists a regular positive cone for F_2. But F_2 cannot admit a regular positive cone.

Short References

[HŠ17]: Hermiller and Šunić. DOI: 10.1142/S0218196717500527.
[Nav11]: Navas. DOI: 10.1016/j.jalgebra.2010.10.020
[Su19]: Su. arXiv: 1905.13001

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