Intro to Left-Orderable Groups

INSTITUTO DE CIENCIAS MATEMÁTICAS

Co-funded by the Horizon 2020 programme
of the European Union

"la Caixa" Foundation

Hang Lu Su
ICMAT-UAM

Structure

The talk will be very introductory.

Part 1: Basics and examples

Part 2: Overview of research.

We will state a lot of results in my research niche, but no proofs of these results.

Part 1: Basics and examples

Left-orderable groups

Def: a group G is *left-orderable* if there exists a total order $\prec$ on the elements of G which is invariant under left-multiplication:

$$g \prec h \iff fg \prec fh, \qquad \forall g, h, f \in G.$$

Why study left-orderable groups: 4 applications

- Used by Mineyev in a proof of the Hanna-Neumann conjecture. Let H and K be non-trivial free groups. Then $\mathrm{rk}(H \cap K) - 1 \leq (\mathrm{rk}(H) - 1)(\mathrm{rk}(K) - 1)$.
- Related to zero divisor conjecture. Let R be a ring without zero divisors and G be torsion free. The conjecture says that the group ring RG has no zero divisors.
  What is known: if G is left-orderable, then RG has no zero divisors.
- Link with dynamics. For a countable group G, G is left-orderable iff it is isomorphic to a subgroup of the set of orientation-preserving homeomorphisms on $\mathbb{R}$.
- Very large classes of groups are orderable: fundamental groups of surface groups are left-orderable except for the fundamental group of the projective plane and RAAGs.

First example

Def: a group G is *left-orderable* if there exists a total order $\prec$ on the elements of G which is invariant under left-multiplication:

$$g \prec h \iff fg \prec fh, \qquad \forall g, h, f \in G.$$

$(\mathbb{Z}, +)$ has a natural left-order given by

$$\cdots < -1 < 0 < 1 < 2 < \dots$$

The order is clearly invariant under addition.

The definition of left-order is completely symmetric and

$$\cdots < 1 < 0 < -1 < -2 < \dots$$

is another left-order.

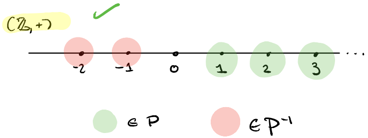
Equivalent notion: positive cone

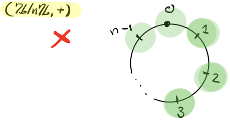
We can look at left-orders in terms of sets called *positive cones*. Roughly speaking, a positive cone is trying to capture the notion of additive positivity in your group.

$P \subset G$ is a *positive cone* for G if

- $PP \subseteq P$ (closed under semigroup operation),
- $G = P \sqcup P^{-1} \sqcup \{1_G\}$ (trichotomy property).

Ex: $(\mathbb{Z}, +)$ admits *a* positive cone. $\mathbb{Z}/n\mathbb{Z}$ does not.





Suppose a group with torsion has a positive cone. Then is some element g such that $g^n = 1$. Then wlog if $g \in P, g^n = 1 \in P$. Contradiction.

Equivalence of positive cone with left-order

Ex: (continued) $(\mathbb{Z}, +)$ admits this natural left-order

$$\cdots < -1 < 0 < 1 < 2 < \ldots$$

The left order is equivalent to defining positive elements

$$P = \{z > 0 \mid z \in \mathbb{Z}\}$$

in the sense that

$$x < y \iff -x + y > 0.$$

Equivalence of positive cone with left-order

The equivalence is easy to prove in general.

Claim: G admits a left-order iff it admits a positive cone.
($\Longrightarrow$) The left-order $<$ defines a positive cone

$$P_< = \{g \in G \mid 1 < g\}.$$

- If $1 < g$ then $g^{-1} < 1$. So $G = P \sqcup P^{-1} \sqcup \{1\}$.
- If $1 < g$ and $1 < h$ then $1 < g < gh$ so $PP \subseteq P$.

Equivalence of a positive cone with left-order

$(\Longleftarrow)$ The positive cone P defines a left-order $<_P$. Indeed, define

$$g <_P h \iff g^{-1}h \in P.$$

This respects left-invariance because $fg <_P fh \iff (fg)^{-1}fh \in P$ but $(fg)^{-1}fh = g^{-1}f^{-1}fh = g^{-1}h \in P \iff g <_P h$.

This is a total order because $g^{-1}h$ is either in P, P^{-1} or is equal to $\{1\}$, in which case $g = h$.

Closure Property 1: subgroups

If $(G, <)$ is a left-orderable group and $H \leq G$, then $(H, <)$ is left-orderable.

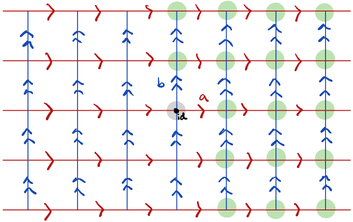
Clearly, if $g < h \iff fg < fh$ for all $g, h, f \in G$, this is also true if $g, h, f \in H$.

ex: $2\mathbb{Z} \leq \mathbb{Z}$.

$$\cdots < -2 < 0 < 2 < 4 < \dots$$

Closure Property 2: extensions

Let's start with an example.
We know that $\mathbb{Z}$ is left-orderable. $\mathbb{Z}^2$ is also left-orderable viewed as $\mathbb{Z} \times \mathbb{Z}$.



A positive cone is for $\mathbb{Z}^2$ is given by

$$P = \{a^n b^m \mid n > 0 \vee n = 0, m > 0\}.$$

This is a lexicographic order with leading factor A, where $A = \langle a \rangle$ because $P = \{z \in \mathbb{Z}^2 \mid \pi_A(z) \in P_A \vee \pi_A(z) = 1_A, \pi_B(z) \in P_B\}$.
$P = P_A B \cup P_B$, where P_A, P_B are positive cones of $\mathbb{Z}$.

Closure 2: extensions

This idea of composing positive cones generalizes to extensions. Suppose you have $1 \to N \to G \to Q \to 1$, with P_N and P_Q.

Then $P_Q N \cup P_N$ is a positive cone for G since:

Every element can be written as $g = qn$, $q \in Q, n \in N$

Trichotomy: g is such that $q \in P_Q, q \in P_Q^{-1}$ or $q = 1$ and similarly for n, so $G = (P_Q N \cup P_N) \sqcup (P_Q N \cup P_N)^{-1} \sqcup \{1\}$.

Semi-group closure: if $g, h \in P$ such that $g = qn, h = pm$ then $gh = qnpm = qp(p^{-1}np)m = qp(n'm)$ where $n' \in N$. Either both q,p are in P_Q or one of $q, p \in P_Q$ and the other is 1, but in both case $qp \in P_Q \implies gh \in P_Q N \subset P$, or $q = p = 1$, in which case $n' = p^{-1}np = n$ and then both $n, m \in P_N$ so $gh \in P_N \subset P$. $\square$

We call this order a *lexicographic order with leading factor Q*.

Applications of extension lemma

Extensions of left-orderable groups are left-orderable. This already gives us quite a lot just knowing that $\mathbb{Z}$ is left-orderable!

Poly-$\mathbb{Z}$ groups are left-orderable.

Recall G is poly-$\mathbb{Z}$ if we have a subnormal series
$G = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = \{1\}$ such that $G_i/G_{i+1} \cong \mathbb{Z}$.
Do induction on the length of the series.

Virtually poly-$\mathbb{Z}$ groups are poly-$\mathbb{Z}$ (Antolín, C. Rivas, and Su, upcoming).

Solvable Baumslag-Solitar groups
$B(1, q) = \langle a, b \mid aba^{-1} = b^q, q \neq 1\rangle$ can be viewed as a semi-direct product of q-adic rationals and integers: $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$.

Locally indicable groups

Def: A group G is *locally indicable* if every non-trivial, finitely generated subgroup has a homomorphism onto $\mathbb{Z}$.

Well-known fact: Locally indicable $\Rightarrow$ left-orderable.

The proof of that fact does not give us an explicit order because it relies on the following equivalence.

G is left-orderable iff for every finite set $\{g_1, \ldots, g_n\}$ of G which does not contain the identity, there exists exponents $\epsilon_i = \pm 1$ such that $1 \notin \langle g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n} \rangle^+$.

Once we have this equivalence, the ϵ_i can be given by the map onto $\mathbb{Z}$.

Next: Sketch proof of equivalence and see where the non-explicit order comes from.

Locally indicable groups

G is left-orderable iff for every finite set $\{g_1, \ldots, g_n\}$ of G which does not contain the identity, there exists exponents $\epsilon_i = \pm 1$ such that $1 \notin \langle g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n} \rangle^+$.

Proof sketch:
$\Longrightarrow$ choose ϵ_i's such that $P \supset \langle g_1^{\epsilon_1}, \ldots, g_n^{\epsilon_n} \rangle^+$ since g_i or $g_i^{-1} \in P$.

$\Longleftarrow$ says that we have a local to global relation: we can chose exponents for finite sets of elements of G to form semigroups excluding 1, and these sets may grow larger and larger - limits are positive cones which live in $\mathcal{P}(G)$, and these limits exist by compactness of $\mathcal{P}(G)$ with product topology. These statements are not super obvious a priori!

All I want you to remember: Locally indicable $\Rightarrow$ left-orderable but no explicit left-order is given.

Applications of local indicability

Free groups are left-orderable.

Proof:

- Nielsen–Schreier theorem: Every subgroup of a free group is free.
- In particular, every finitely generated subgroup of a free group is free.
- For each f.g. subgroup, get homomorphism onto $\mathbb{Z}$ by sending every generator to 1.
- F is locally indicable, thus left-orderable.

Explicit orders can be given by the Magnus embedding and an action on a tree (Dicks and Z. Sunic, 2020). There are uncountably many left-orders.

RAAGs are poly-free, where the length of the subnormal series is bounded by the chromatic number of the defining graph (S. Hermiller and Z. Sunic, 2007). Using induction and extension lemma we get RAAGs are left-orderable.

Surface groups

Surface groups are left-orderable apart from the fundamental group of the projective plane.

The proof is *long* and uses a lot of algebraic topology which I will not explain due to lack of time. It comes down to the following.

- Closed surfaces (except from T^2, $P^2, P^2 \# P^2$) are a cover of $3P^2 = P^2 \# P^2 \# P^2$.
- If S is such a closed surface, then $\pi_1(S) \leq \pi_1(3P^2)$
- left-orders are closed under subgroups, so we restrict our attention to $\pi_1(3P^2)$.
- Use the fact that $3P^2$ is homeomorphic to $P^2 \# T^2$ to get a short exact sequence

  $$1 \to \pi_1(\widetilde{3P^2}) \to \pi_1(3P^2) \to \mathbb{Z}^2 \to 1$$

  where $\pi_1(\widetilde{3P^2})$ is free and thus left-orderable.
- Use the extension lemma to obtain that $\pi_1(3P^2)$ is left-orderable.

Surface groups

So far: we have vaguely shown that any closed surface which is not $P^2, P^2 \# P^2$ and T^2 is left-orderable by reducing it to $\pi_1(3P^2)$ is left-orderable.

The rest of the groups that are not *covered* are $\pi_1(P^2) = \mathbb{Z}/2\mathbb{Z}$ which we know not to be left-orderable, $\pi_1(P^2 \# P^2) = K_2$ and $\pi_1(T^2) = \mathbb{Z}^2$ which we know to be left-orderable. (We will give a postive cone for K_2 later).

Dynamic realization of left-orderable groups

Claim: For a countable group G, G is left-orderable iff it is isomorphic to a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$.

Proof sketch:
($\Rightarrow$) Let $t : G \to \mathbb{R}$ be an injective map such that $g \prec h \implies t(g) < t(h)$. Define action of G on $t(G)$ equivariantly $g(t(h)) = t(gh)$. Extend action on $\mathbb{R} - t(G)$ by defining the action affinely: if $x \in (t(g), t(h))$ then $x = at(g) + (1-a)t(h)$ and apply action linearly.

($\Leftarrow$) Order group elements by their action on a countable dense set. $f \prec g \iff f(x_i) < g(x_i)$ for x_i of minimal index in the countable dense set such that $f(x_i) \neq g(x_i)$.

End of basics.

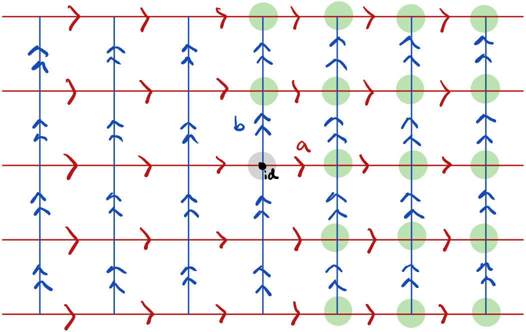
Recommended reference: *Ordered Groups and Topology* by Adam Clay and Dale Rolfsen.

Any questions?

Part 2: Introduction to our research

Remark on finite generation

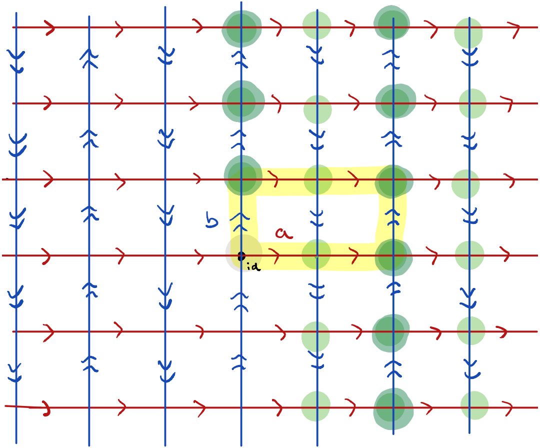
Recall that a positive cone for $\mathbb{Z}^2$ was given by two finitely generated positive cones, $P_A = \langle a \rangle^+$, $P_B = \langle b \rangle^+$ and $P_A B \cup P_B$ is a positive cone for $\mathbb{Z}^2$.



In some sense it's as easy to know if something is positive in $\mathbb{Z}^2$ as it was for $\mathbb{Z}$, but this positive cone for $\mathbb{Z}^2$ is *not* finitely generated.

Remark on finite generation

In fact, $\mathbb{Z}^2$ also inherits an isomorphic positive cone as a subgroup of index two in $K_2 = \langle a, b \mid a^{-1}bab = 1\rangle$ with positive cone $\langle a, b\rangle^+$.



Again, the inherited positive cone for $\mathbb{Z}^2$ is *not* finitely generated.

Remark on finite generation

Being finitely generated as a positive cone

- is not closed under taking extensions.
- is not closed under taking finite index subgroups.

Takeaway: being finitely generated is not a very good property to describe how hard it is to determine whether something is in a positive cone.

Research motivation: have a good way to describe positive cone complexity.

Next: Let's prove that $\mathbb{Z}^2$ has *no* finitely generated positive cone.

Topology of the Space of Left-Orders

Take LO(G) to be the spaces of left orders G
(Recall: a left-order is equivalent to a positive cone, so points in that space are positive cones).

- All $P \in \mathrm{LO}(G)$ are subsets of $G - \{1_G\}$,
  $\implies$ LO(G) is a subset of the power set of $G - \{1_G\}$.
- LO(G) inherit the natural product topology on $\subseteq \{+1, -1\}^{G-\{1\}} \simeq \mathcal{P}(G - \{1_G\})$.

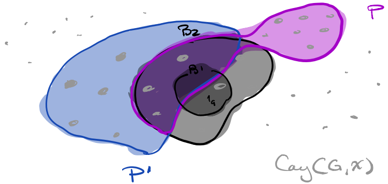
Every element P has $|G| - 1$ coordinates indexed by $g \neq 1_G$.

$$\begin{cases} \pi_g(P) = +1, & g \in P \\ \pi_g(P) = -1, & g \in P^{-1} \end{cases}$$

Ex: LO($\mathbb{Z}$) has two points, name them P and P'. They are defined by $\pi_1(P) = 1$ or $\pi_{-1}(P') = 1$; whether you choose 1 or -1 to be in your positive cone.

Topology of Space of Positive Cones

We say that two positives cones are close in that space if they agree on large balls of group elements based at 1_G.



The point: If P is finitely generated then for a sufficiently large ball B_n, P will be the only positive cone which agrees with itself on B_n because B_n contains all the generators, and any semigroup containing all these generators take up about half of G, so P is an isolated point.

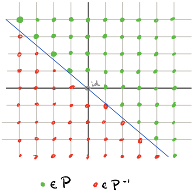
Ex: in $\mathbb{Z}$, P, P' are isolated because they are generated by 1 and -1, and can't agree with each other on a single element.

P finitely generated $\implies$ P is an isolated point.

Positive cones of $\mathbb{Z}^2$

Claim: We will show that $LO(\mathbb{Z}^2)$ has no isolated points, and thus no finitely generated positive cones.

The positive cones in $\mathbb{Z}^2$ are defined by the half-spaces and a choice of orientations. This half-space represents a point in LO($\mathbb{Z}^2$).

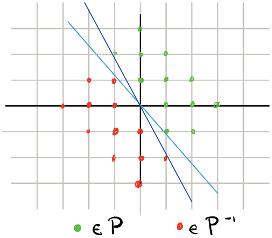


- The half-space is clearly closed under semi-group operation (adding two vectors together).
- The half-space clearly partitions $\mathbb{Z}^2$ appropriately.

Neighbourhoods of LO($\mathbb{Z}^2$)

Claim: We will show that $LO(\mathbb{Z}^2)$ has no isolated points, and thus no finitely generated positive cones.

A *neighbourhood* in LO($\mathbb{Z}^2$) is the set of all positive cones agreeing on a finite ball, but which do not necessarily coincide outside of the ball.



You can draw two different such slopes for any finite ball!

$\implies$ LO($\mathbb{Z}^2$) has no isolated points.

$\implies$ $\mathbb{Z}^2$ has no finitely generated positive cones! $\square$

Research motivation

What we just saw: That $\mathbb{Z}^2$ has no finitely generated positive cones.
But

- $\mathbb{Z}^2$ is an extension of two groups with finitely generated positive cones.
- $\mathbb{Z}^2$ is a subgroup of finite index in a group which has a finitely generated positive cone. (In fact, the Klein Bottle group has *only* finitely generated positive cones).

Therefore, finite generation doesn't capture positive cone complexity well.

Research motivation: have a good way to describe positive cone complexity.

We study positive cones and regular languages.

Positive cone language

Let $G = \langle X \mid R \rangle$ be finitely generated.

Let $X^* := \bigcup_{n=0}^{\infty} X^n$.

A *language* is a subset of $L \subset X^*$, and elements of L are called *words*.

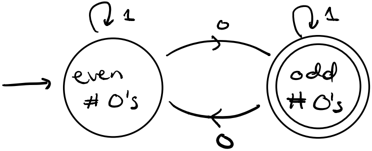
We say that L *evaluates* to P if there is an evaluation map $\pi : X^* \to G$ such that $\pi(L) = P$. If P is a positive cone, we call L a *positive cone language*.

Regular languages

Def: A regular language is a language accepted by a *finite state automaton*.

Finite state automata capture the idea of needing *finite memory*.

Ex 1: The language accepted by this automaton is the set of binary strings with an odd number of 0's.

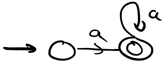


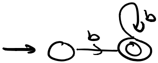
Ex of accepted strings: $10, 000, 01$. Non-examples: $\varepsilon, 1, 00$.

Regular languages are the simplest languages in a classification of languages called the *Chomsky hierarchy*.

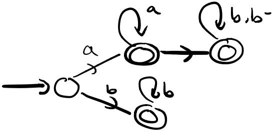
Positive cone of $\mathbb{Z}^2$ as a regular language

Ex: $\mathbb{Z}^2 = A \times B$.
$P_A = \langle a \rangle^+$, $B = \langle b \rangle$,
$P_{\mathbb{Z}^2} = P_A B \cup P_B = \{a^n b^m \mid n > 0\} \cup \{b^m \mid m > 0\}$.

P_A :


P_B :


B :


$P_A B \cup P_B$:


Overview of research niche

Closure properties of regular positive cones

- Regular positive cones are closed under extensions. Proof is really easy!
- Regular positive cones are closed under finite index subgroups. (Su, 2020).
- Having a regular positive cone is independent of generating sets, assuming the sets are finite. (Known fact and easy to proof).
- However, it is positive cone dependent. $B(1, q)$ where $q > 1$ has both regular and non-regular positive cones (Antolín, C. Rivas, and Su, upcoming).

Overview of research niche II

Geometric property of regular positive cones

Def: A set is $P \subseteq G$ is *coarsely connected* if it is connected in the Cayley graph up to some R-neighbourhood, for $R \geq 0$.

- Regular positive cone $P \implies$ the set P is coarsely connected (Alonso, Antolin, and C. Rivas, 2020).

Applications:

- Non-abelian free groups have no coarsely connected positive cones and hyperbolic groups with coarsely connected positive cones have to be very distorted in the sense of not being connected by quasi-geodesics (Alonso, Antolin, and C. Rivas, 2020).
- Relatedly, free products have no regular positive cones (S. M. Hermiller and Zoran Sunic, 2017) and acylindrically hyperbolic groups have no regular quasi-geodesic positive cones (Su, 2020).

Overview of research niche

Crossing left-orderable groups with $\mathbb{Z}$:
Something inherent about positive cones change when you cross groups with $\mathbb{Z}$.

- Let A, B be left-orderable. $A * B$ has no isolated orders (Deroin, Navas, and C. Rivas, 2014). In particular F_2 has no isolated orders.
- $F_2 \times \mathbb{Z}$ has both isolated and non-isolated orders (Mann and Cristobal Rivas, 2018).
- Moreover, $F_2 \times \mathbb{Z}$ has a finitely generated positive cone, which is part of a new infinite family of groups with k-finitely generated positive cones for any $k \geq 3$ (Su, 2020). Whether such a family existed was a question left open by Navas, 2011.
- In general, free products of groups with regular positive cones $A * B$ have a one-counter order (Dicks and Z. Sunic, 2020) but no regular order (S. M. Hermiller and Zoran Sunic, 2017).
- $(A * B) \times \mathbb{Z}$ has a regular order (Antolín, C. Rivas, and Su, upcoming).

Advertisement

If any of this has intrigued you:

- I am giving a less introductory talk at **Queen's University Nov 26, 2pm EST (7pm UK time?)** on recent results.
  https://www.queensu.ca/mathstat/seminars/dynamics
- A **past talk** on my first paper is available in video format on my website **homeowmorphism.com/**
  There's many details and pictures; the talk is fairly accessible!
- I post **recordings** of my talk on my websites in general, including this one and the upcoming one.
- I may post **more details** I skipped over for the sake of time, like really proving that surface groups minus the projective plane are left-orderable. I learned a lot myself preparing this talk :).

Bibliography I

Alonso, J., Y. Antolin, and C. Rivas (2020). "On the geometry of positive cones in finitely generated groups". In: *ArXiv e-prints*. arXiv: 2001.10286.

Antolín, Y, C. Rivas, and H.L. Su (upcoming). "Left-orders of low language complexity". In: *upcoming*.

Deroin, B., A. Navas, and C. Rivas (2014). *Groups, Orders, and Dynamics*. arXiv: 1408.5805.

Dicks, W. and Z. Sunic (2020). "Orders on trees and free products of left-ordered groups". In: *Canadian Mathematical Bulletin* 63 (2), pp. 335–347.

Hermiller, S. and Z. Sunic (2007). " Poly-free constructions for right-angled Artin groups ". In: *Journal of Group Theory* 10 (1).

Hermiller, Susan M. and Zoran Sunic (2017). "No positive cone in a free product is regular". In: *IJAC* 27, pp. 1113–1120.

Bibliography II

Mann, Kathryn and Cristobal Rivas (2018). "Group orderings, dynamics and rigidity". In: *Ann. Inst. Fourier* 68 (4), pp. 1399–1445.

Navas, A. (2011). "A remarkable family of left-ordered groups: Central extensions of Hecke groups". In: *Journal of Algebra* 328 (1), pp. 31–42.

Su, H.L. (2020). "Formal language convexity in left-orderable groups". In: *Internat. J. Algebra Comput* 30 (07), pp. 1437–1456.