Formal Languages in Left-Orderable Groups

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Left-orderable groups

Def: a group G is *left-orderable* if there exists a total order $<$ on the elements of G which is invariant under left-multiplication:

$$g < h \iff fg < fh, \qquad \forall g, h, f \in G.$$

Why study left-orderable groups:

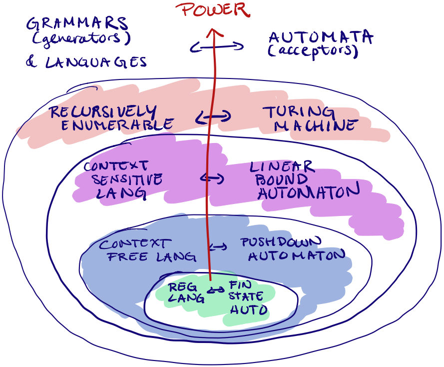
- Used in proof of Hanna-Neumann conjecture. Let H and K be non-trivial free groups. Then $\mathrm{rk}(H \cap K) - 1 \leq (\mathrm{rk}(H) - 1)(\mathrm{rk}(K) - 1)$.
- Link with dynamics. G is left-orderable if and only if there is a map from G to the set of orientation-preserving homeomorphisms on a totally ordered space (if G is countable, the space is $\mathbb{R}$).

Goal

Goal of PhD (roughly speaking): Given a group which is left-orderable, can you come up with explicit left-orders that are optimally easy to compute?

Goal

More formally: The goal is to classify left-orders of groups with respect to some minimal measure of computational complexity, called the *Chomsky hierarchy*. We will be applying the measure to *formal language representations* of positive cones.



Good news: you don't have to know what any of this means! :)

How to define complexity measures on left-orders

Positive cones?! Formal language representations? I thought we were talking about orders!

Question: How do we put a formal language (or measure of computational complexity) on a left-order?

Answer: We will use languages (set of words) which represent *positive cones* to classify the complexity of left-orders. Positive cones are subsets of your left-orderable groups, and defining a positive cone is the same as defining a left order!

The goal is to be able to read a class of words (for example, reduced words) and decide whether that word is in your positive cone or not.

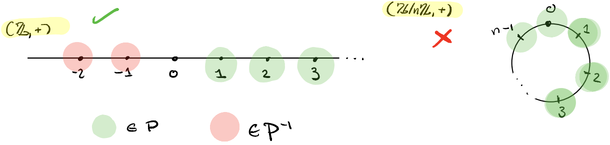
Positive cones

Roughly speaking, a positive cone is trying to capture the notion of additive positivity.

$P \subset G$ is a *positive cone* for G if

- $PP \subseteq P$ (closed under semigroup operation),
- $G = P \sqcup P^{-1} \sqcup \{1_G\}$ (trichotomy property).

Ex: $(\mathbb{Z}, +)$ admits *a* positive cone. $\mathbb{Z}/n\mathbb{Z}$ (or any group with torsion) does not.



Admitting a left-order and admitting a positive cone are equivalent.

Equivalence with left-orderability

Def: a group G is *left-orderable* if there exists a total order $<$ such that for all elements $g, h, f \in G$,

$$g < h \iff fg < fh.$$

Claim: G admits a left-order iff it admits a positive cone.
$(\implies)$ The left-order $<$ defines a positive cone

$$P_< = \{g \in G \mid 1 < g\}.$$

$(\impliedby)$ The positive cone P defines a left-order $<_P$

$$g <_P h \iff g^{-1}h \in P.$$

Back to ex: $(\mathbb{Z}, +)$ admits left-order $\cdots < -1 < 0 < 1 < 2 < \ldots$ which is can be defined by positive cone: $x < y \iff -x + y > 0$.

Toy Example: K_2 and $\mathbb{Z}^2$

Take the Klein bottle group given by $K_2 = \langle a, b : a^{-1}bab = 1\rangle$.

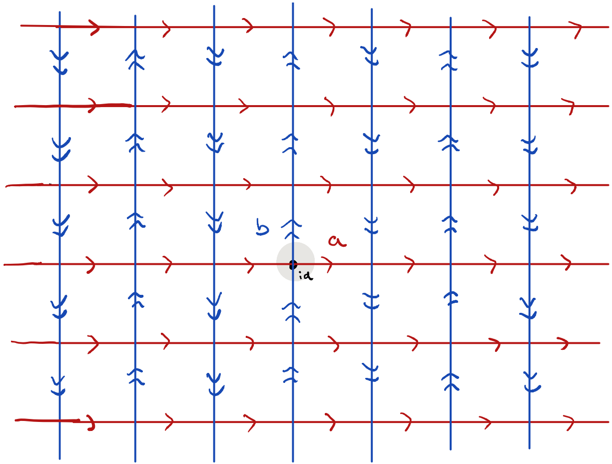
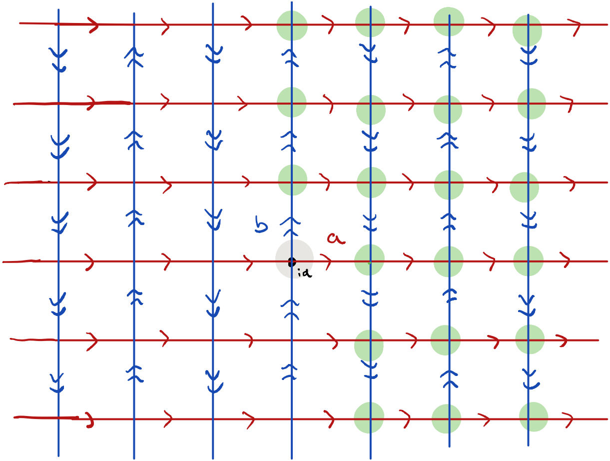


Fig: Cayley graph of K_2.

Toy Example: K_2 and $\mathbb{Z}^2$

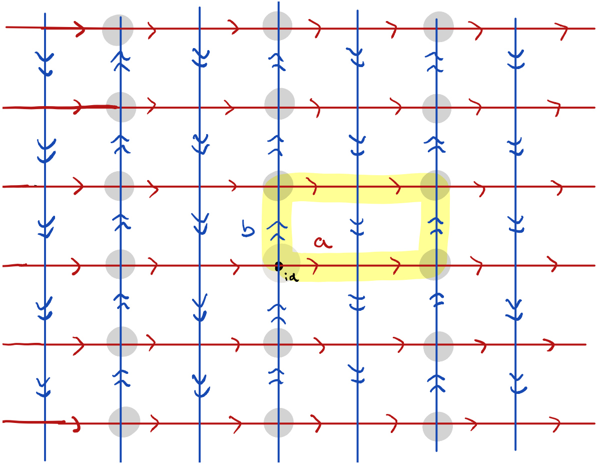
K_2 has a positive cone defined by $P = \langle a, b \rangle^+$, represented by green dots.



Clearly, P is a semi-group partitioning K_2 into $P \sqcup P^{-1} \sqcup \{1\}$.

Toy Example: K_2 and $\mathbb{Z}^2$

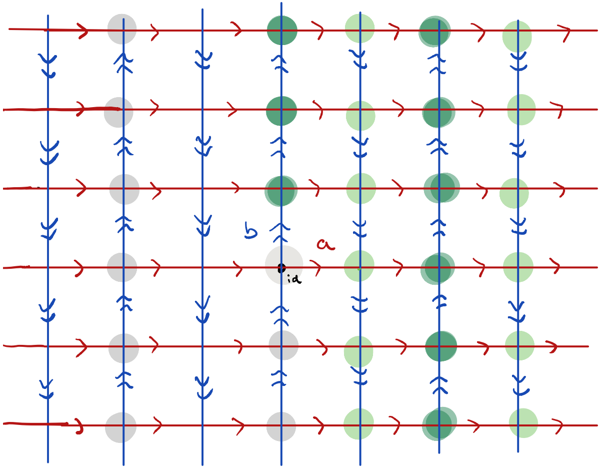
K_2 has index two subgroup $\mathbb{Z}^2 = \langle a^2, b \rangle$, represented by grey dots. The relation is represented by highlighted part.



What happens when we put the two pictures together?

Toy Example: K_2 and $\mathbb{Z}^2$

$P \cap \mathbb{Z}^2$ is a positive cone for $\mathbb{Z}^2$, represented by dark green dots.



This is non-specific to the example. If $H \subseteq G$ and P is a positive cone for G, then $P \cap H$ is a positive cone for H.

Toy Example: K_2 and $\mathbb{Z}^2$

Reminder: The group K_2 had a nice description of the positive cone P as a finitely generated semigroup.

Question: Can we also describe $P \cap \mathbb{Z}^2$ in an equally nice way (as finitely generated semigroup)?

Answer: No (surprisingly?!)

Next: Let's get some intuition for why $\mathbb{Z}^2$ does not admit *any* finitely generated positive cones.

Topology of the Space of Left-Orders

Take $\mathrm{LO}(G)$ to be the spaces of left orders G
(Recall: a left-order is equivalent to a positive cone).

- All $P \in \mathrm{LO}(G)$ are subsets of $G - \{1_G\}$,
  $\implies \mathrm{LO}(G)$ is a subset of the power set of $G - \{1_G\}$.
- $\mathrm{LO}(G)$ inherit the natural product topology on $\subseteq \{+1, -1\}^{G-\{1\}} \simeq \mathcal{P}(G - \{1_G\})$.

Every element P has $|G| - 1$ coordinates indexed by $g \neq 1_G$.

$$\begin{cases} \pi_g(P) = +1, & g \in P \\ \pi_g(P) = -1, & g \in P^{-1} \end{cases}$$

Topology of Space of Positive Cones

The topology is metrizable for countable groups (in this talk, we only care about finitely generated groups which are countable).

Metric: If G is countable, then G admits a complete exhaustion by finite sets $\{1_G\} = B_0 \subset B_1 \subset \dots$. We can define the distance between two points $P, P' \in \mathrm{LO}(G)$ by

$$d(P, P') = 2^{-n},$$

where n is the maximal non-negative integer such that P and P' coincide on B_n. The more elements P, P' have in common the closer they are.

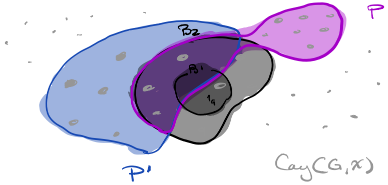


Fig: here P, P' coincide at B_2, but not outside, so their distance is $1/4$.

Topology of Space of Positive Cones

Two positives cones are close in that space if they agree on large balls of group elements based at 1_G.

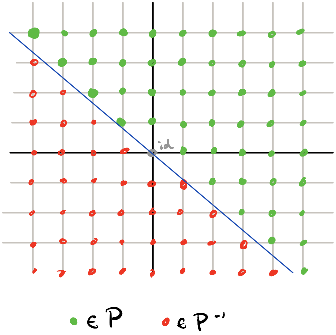
Point: If P is finitely generated then for a sufficiently large ball B_n, P will be the only positive cone which agrees with itself on B_n because B_n contains all the generators, and any semigroup containing all these generators take up about half of G, so P is an isolated point.

P finitely generated $\implies$ P is an isolated point.

Next: We will show that $LO(\mathbb{Z}^2)$ has no isolated points, and thus no finitely generated positive cones.

Positive cones of $\mathbb{Z}^2$

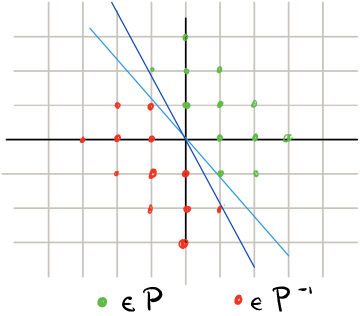
The positive cones in $\mathbb{Z}^2$ are defined by the half-spaces and a choice of orientations. This half-space represents a point in $\mathrm{LO}(\mathbb{Z}^2)$.



- The half-space is clearly closed under semi-group operation (adding two vectors together).
- The half-space clearly partitions $\mathbb{Z}^2$ appropriately.

Neighbourhoods of LO($\mathbb{Z}^2$)

A *neighbourhood* in LO($\mathbb{Z}^2$) is the set of all positive cones agreeing on a finite ball, but which do not necessarily coincide outside of the ball.



You can draw two different such slopes for any neighbourhood of LO($\mathbb{Z}^2$).
$\implies$ LO($\mathbb{Z}^2$) has no isolated points.

$\implies$ $\mathbb{Z}^2$ has no finitely generated positive cones! $\square$

Result 1: Passing to Finite Index Subgroups

Review:

- We had K_2, a group with a finitely generated positive cone.
- We had $\mathbb{Z}^2 \leq K_2$, a subgroup of index 2 with no finitely generated positive cone.

Takeaway: Being finitely generated as a positive cone is not passed down to finite index subgroups!

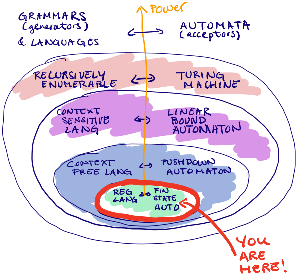
Question: What property of positive cones pass to finite index subgroups?

Result 1: Passing to Finite Index Subgroups

Theorem 1 [?]

If G is a finitely generated group with positive cone P and regular language representing P and H is a finite index subgroup of G, then H has a regular language representing $P \cap H$.

By the toy example with $\mathbb{Z}^2 \leq K_2$, this is the minimal complexity that is passed down to finite index. (finitely generated $\implies$ regular)



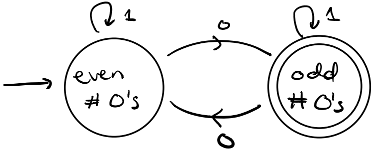
Next: But what *are* regular languages?

Regular languages

Def: A regular language is a language accepted by a *finite state automaton*.

Finite state automata capture the idea of needing *finite memory*.

Ex 1: The language accepted by this automaton is the set of binary strings with an odd number of 0's.



Ex of accepted strings: $10, 000, 01$. Non-examples: $\varepsilon, 1, 00$.

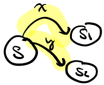
Regular languages

Instructions to obtain accepted language:

The accepted language is a subset of all the words you may form using the automaton.

1. Starting with an empty word, start at the state which has an arrow from nothing pointing to it (*start state*).

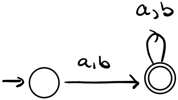
2. Read the arrows labeled by a letter in the alphabet, and add this letter to the word you are forming.



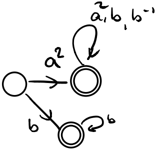
3. The language accepted is the set of words spelled by the arrows such that the last arrow leads to a *final state*.

Regular languages

Ex 2: The language accepted by the automaton below is the entire semigroup $\langle a, b\rangle^+$ of positive words in a and b. This evaluates to a positive cone P for K_2. (All finitely generated positive cones are regular by the same reasoning.)



The language accepted by the automaton below is $\{a^{2n}b^m \mid n \in \mathbb{Z}^+, m \in \mathbb{Z}\} \cup \{b^m \mid m \in \mathbb{Z}^+\}$. This evaluates to $P \cap \mathbb{Z}^2$.



Point: This is a concrete example of regular positive cones passing to finite index subgroups.

Aside: words vs group elements

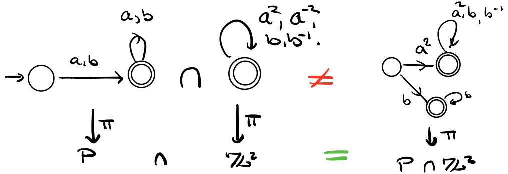
We want to show the following.

Theorem 1

For finitely generated groups, regular positive cones pass to finite index subgroups.

Obvious but doesn't work:

(regular positive cone language) $\cap$ {all words generators of subgroup}



Ex: For K_2, P has no b^{-1} but $a^2 b^{-1}$ is accepted for $P \cap \mathbb{Z}^2$.

We are interested in inheriting the positive cone at the language level!

Aside: words vs group elements

Important distinction: It is important to distinguish between a word and the group element it represents (it's π-image). For example, $\pi(b^{-1}b) = \pi(bb^{-1}) = 1_G$, but $b^{-1}b \neq bb^{-1}$ at the level of words.

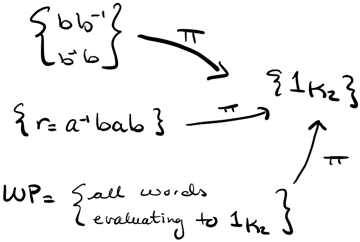


Fig: different languages (sets of words) which evaluates to the identity. WP stands for Word Problem, and WP is a superset of all such languages. Figuring out WP for various groups is a famous computational problem in geometric group theory.

Aside: words vs group elements

Similarly, given a positive cone, there are many languages which surjects to it. We call these languages *positive cone languages*.

$$\begin{tikzcd}
L = \langle a, b \rangle^{+} \arrow[dr, "\pi"] & \\
& P \\
L' \arrow[ur, "\pi"] &
\end{tikzcd}$$

Link to co-word problem: if a word is in a positive cone language (or inverse positive cone language) then it is *not* equal to the identity.

Aside: words vs group elements

Just like in the word problem, the complexity of a positive cone language can tell us about its geometry!

Def: A subset $P \subseteq G$ is *coarsely connected* if there is an $R \geq 0$ such the R-neighborhood of P is connected.

Link to geometry: regularity of a positive cone language $L \implies$ the positive cone is coarsely connected [AAR20].

Back to problem

For a subgroup of finite index H, we are interested in a language which surjects to a positive cone for $H \cap P$, constructed from a regular language which surjects to P.

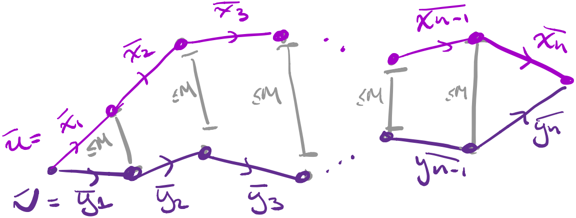
How do we construct such a language for $H \cap P$?

Idea: how about we design an automaton which 'keeps up with' the arrows of the positive cone automaton for the group?

Fellow-travel

Speaking of keeping up with... Have you heard of ~~Keeping Up With the Kardashians~~ fellow-travel?

Let $\bar{u} := \pi(u)$, where u corresponds to a path in the Cayley graph and $\bar{u}$ the group element (end vertex of path).



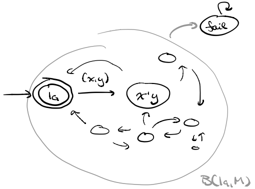
Def: Let $u = x_1 \ldots x_n$ and $v = y_1 \ldots y_n$. Let $u_i = x_1 \ldots x_i, v_i = y_1 \ldots y_i$ be the ith prefixes. We say that u and v *synchronously* M-fellow-travel if $d(\bar{u}_i, \bar{v}_i) \leq M$ for $1 \leq i \leq n$ and (for this talk) $\bar{u} = \bar{v}$.

Fellow-travel

Fact: The language of synchronously M-fellow-travelling words is regular.

Key: use finite memory to remember the difference between words.

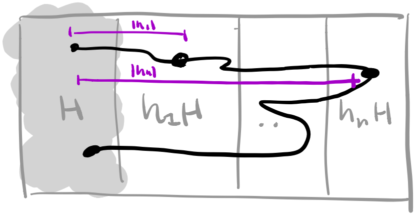
- States: all group elements which are in a ball of radius M centered at identity, plus fail state.
- Transitions: going from one difference to another. If the pair of paths goes from (u, v) to (ux, vy) then the difference goes from $\bar{u}^{-1}\bar{v}$ to $x^{-1}\bar{u}^{-1}\bar{v}y = (\bar{u}\bar{x})^{-1}(\bar{v}\bar{y})$. If $(\bar{u}\bar{x})^{-1}(\bar{v}\bar{y})$ is not in ball of radius M, then transition to fail state.
- Start state: the identity. Final state: the identity.



Proof idea of regularity passing to finite index

Steps of proof : Assume $G = \langle X \mid S \rangle$ finitely generated.

1. Let H be a finite index subgroup. Then H is *L-convex* for any language $L \subset X^*$, meaning that for any language L, the path induced by a word with endpoints in H is point-wise $\leq R$ away from H. (Similar to quasi-convexity but w.r.t. L).

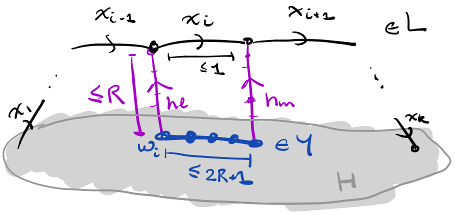


Illustrated: Let $h_1, \ldots, h_n$ be coset representatives of H. Then every element of G is at distance $\leq \max_{i=1}^n |h_i|$ of H. Set $R := \max_{i=1}^n |h_i|$.

Proof idea of regularity passing to finite index

2. H L-convex $\implies$ any word $u \in L$ such that $\bar{u} \in H$ can be finitely generated by

$$Y = \{w \in X^* : \bar{w} \in H, \quad |w| \leq 2R + 1\}.$$

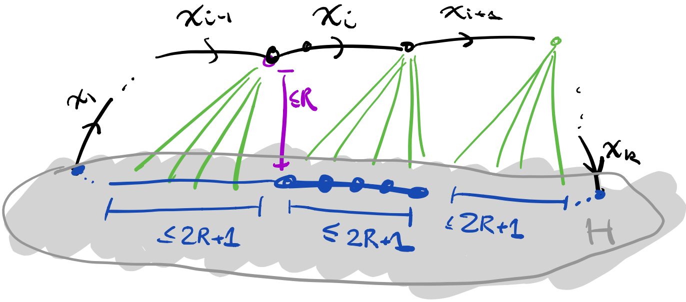


w_i is a geodesic connecting the two projected endpoints of x_i, and $\bar{w}_i \in H$ because its endpoints are in H.

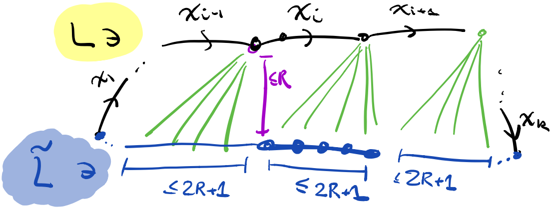
Note: $Y^* \not\subset L \implies$ need more restrictions.

Proof idea of regularity passing to finite index

3. **Observation:** Wait, top and bottom words *almost* fellow-travel if the top word waits for the bottom $\implies$ we call this *asynchronously fellow-travelling*. The parameter here if $(3R+1)$ by triangle inequality.



Proof idea of regularity passing to finite index



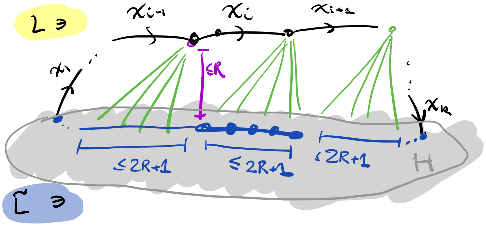
Proposition [Su2019]:

- The language of pairs of words which $(3R+1)$-asynchronously fellow-travel is regular (by adding padding).
- We can set the top language to be a fixed regular positive language L and the language of pairs of words will stay regular (Prop. calc).
- The bottom language by itself is also regular (Prop. calc).

Call the obtained bottom language $\tilde{L}$. Since $\pi(L) = P, \pi(\tilde{L}) = P$.
$\implies$ we just have to add H-restriction back to bottom language!

Proof idea of regularity passing to finite index

Recall: $Y = \{w \in X^* : \bar{w} \in H, \quad |w| \leq 2R+1\}$, but Y^* had potentially 'too many words' (words which do not evaluate necessarily evaluate to P).



4 $\implies$ $\tilde{L} \cap Y^*$ is the desired language.

- Every element in $P \cap H$ has a representative in $\tilde{L} \cap Y^*$ by construction.
- Every word $\tilde{L} \cap Y^*$ corresponds to an element of $P \cap H$ because $\pi(\tilde{L}) = P$ and $\pi(Y^*) \subseteq H$. $\square$

You've worked hard!

Next: just one application on this.

Now is a good time to ask questions!

Application of regularity passing to finite index

Observation: We didn't use finite index directly to show regularity passing to finite index subgroups - we used L-convexity. Here's an application of the proof of Theorem 1.

Application 1 [?]

A quasi-geodesic positive cone language of a finitely generated acylindrically hyperbolic group cannot be regular.

Facts:

1. F_2 is a subgroup of any f.g. acylindrically hyperbolic group. F_2 embeds in a 'nice' way with quasi-geodesics of G (is *hyperbolically embedded* [DGO17]). $\implies$ [Sis16] F_2 is L-convex in G for any quasi-geodesic language L.
2. F_2 cannot have a regular positive cone [HS17].

Proof of Theorem 1 implies: if G has a regular quasi-geodesic positive cone $\implies$ F_2 has a regular positive cone. Contradiction. $\square$

We did this!

Let's take a deep breath.

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I ♡ f.g. positive cones. They are easy.

Let's only talk about them from now on.

PS: if you got lost (as I do), this is a good time to start listening again.

Result 2: Constructing an infinite family of groups with finitely generated positive cones

Problem: we don't have many examples of groups with f.g. pos. cones!

Theorem [Nav11]

There is an infinite family of groups with 2-generated positive cones, $\Gamma_n = \langle a, b \mid ba^n b = a \rangle$ with positive cones $P_n = \langle a, b \rangle^+$ for $n \in \mathbb{N}$.

Ex: $\Gamma_1 = K_2$, the Klein bottle group from before!

Result 2: Constructing an infinite family of groups with finitely generated positive cones

Navas' question: For every $k \geq 3$, find an infinite family of groups which admit a positive cone that is k-generated.

k-**generated:** A positive cones which can be generated by k elements but cannot be generated by less than k elements.

We answer Navas' question completely.

Theorem 2 [?]

For every integer $m \geq 2$, and integer $n \geq 2$ of the form $n = m - 1 + mt$ for some odd integer t, there is a subgroup of index m in Γ_n which admits a positive cone of rank $m + 1$.

Result 2: Constructing an infinite family of groups with finitely generated positive cones

Strategy: Do a similar 'projecting to subgroup' strategy that we did for Theorem 1, but the strategy becomes easier in this case! It is a sort of Reidemeister-Schreier method type argument.

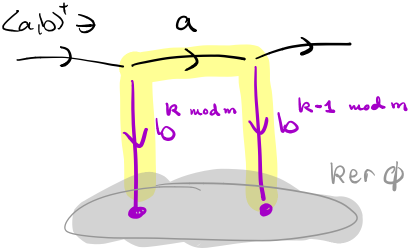
For every m, pick n such that

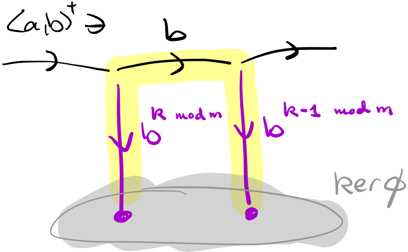
$$\phi : \Gamma_n \to \mathbb{Z}/m\mathbb{Z},$$

$$a \mapsto 1, \quad b \mapsto 1$$

is a homomorphism. By the relation $ba^n b = a$, this means that $n + 1 = 0$ mod m
$\implies n = m - 1 + mt$ for integer t.

Result 2: Constructing an infinite family of groups with finitely generated positive cones





Since $b \mapsto 1 \in \mathbb{Z}/m\mathbb{Z}$, this means that

$$T = \{b^0, b^{-1}, \dots, b^{-(m-1)}\}$$

is a transversal (set of coset representatives) for $\ker\phi$.

For the right choice of $b^{-k} \in T$,

$$b^{-k}ab^{k-1} \text{ and } b^{-k}bb^{k-1} \in \ker\phi.$$

Result 2: Constructing an infinite family of groups with finitely generated positive cones

The magic:

- By the relation in Γ_n, $b^{-k}ab^{k-1} \in P$ for all $k \geq 1$ (in the proof of Theorem 1, our generators did *not* necessarily project to positive cone elements).
- You get finitely many $b^{-k}ab^{k-1}$'s. Moreover, $b^{-k}bb^{k-1}$ is either b^m or 1.

As a result,

$$Y = \{b^{-s}ab^{s-1}\}_{s=1}^{m-1} \cup \{b^m\}$$

generates all of $\ker \phi \cap P = \langle Y \rangle^+$.

Recap: We've shown that the rank of P is $\leq m + 1$ since $|Y| = m + 1$. Next, we need to show that the rank is equal to $m + 1$. We will use the Reidemeister-Schreier method.

Result 2: Constructing an infinite family of groups with finitely generated positive cones

The Reidemeister-Schreier method [LS01] - it's an algorithm:
Input: a finite presentation for G and a finite index subgroup H with a choice of transversal.

Output: a finite presentation for H.

Point: Y coincides with the finite generating set outputted for the Reidemeister Schreier method on Γ_n with finite index subgroup $\ker\phi$ and choice of transversal $T = \{b^0, \ldots, b^{-(m-1)}\}$.

Proving rank of $P \cap H$ is $m+1$:

- Continue using Reidemeister-Schreier to get relations for $\ker\phi$, then abelianize presentation.
- Using $\mathbb{Z}$-module algebra, abelianization has rank $m+1$ when $n = m-1+mt$ for t odd.
- $m+1 = |Y| \geq \mathrm{rk}(P \cap \ker\phi) \geq \mathrm{rk}(\ker\phi) \geq \mathrm{rk}(\mathrm{Ab}\ker\phi) = m+1$. $\square$

I'm almost done now.

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One last application.

Application of Result 2: $F_2 \times \mathbb{Z}$

Let's look at $F_2 \times \mathbb{Z}$ - why do we care?

- Hermiller and Sunic found in 2017 that F_2 does not admit a regular positive cone [HS17].
- $F_2 \times \mathbb{Z}$ was found to have isolated points in its space of left-orders by Mann and Rivas [MR18].
- Recall: finitely generated pos. cone $\implies$ isolated points in space of left-orders.
- **Question:** Is the converse true in this case?
- $F_2 \times \mathbb{Z}$ is a subgroup of index 6 in $\Gamma_2 \implies$ it has a regular positive cone...promising! (This has been prior discovered by C. Rivas in 2019).

Application 2 [?]

$F_2 \times \mathbb{Z}$ has a finitely generated positive cone.

Application of Result 2: $F_2 \times \mathbb{Z}$

Application 2 [?]

$F_2 \times \mathbb{Z}$ has a finitely generated positive cone.

Fact: $F_2 \times \mathbb{Z} \cong \ker \phi'(\Gamma_2)$ where
$\phi' : \Gamma_2 \to \mathbb{Z}/6\mathbb{Z}, \quad \phi'(a) = 4, \phi'(b) = 1.$

In general,

$$Y = \{b^{-s}ab^{s+(m-\phi'(a))}\}_{s=0}^{\phi'(a)-1} \cup \{b^{-s}ab^{s-\phi'(a)}\}_{s=\phi'(a)}^{m-1} \cup \{b^m\}.$$

$\implies$ The rank of $P \cap (F_2 \times \mathbb{Z})$ is at most 7 (actually can get ≤ 6).

Point: In this case, n and m are not as in Theorem 2, so we can't get an exact rank on the positive cone, but we can still say it's finitely generated!

Thank you!

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