Formal languages and Left-Orderable Groups II

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Left-orderable groups

Def: a group G is *left-orderable* if there exists a total order $\prec$ on the elements of G which is invariant under left-multiplication:

$$g \prec h \iff fg \prec fh, \qquad \forall g, h, f \in G.$$

Why study left-orderable groups: 4 applications

- There is a large class of groups which are left-orderable. All surface groups except the one for the projective plane are left-orderable. RAAGs and virtually poly-$\mathbb{Z}$ are left-orderable.
- Link with dynamics. For a countable group G, G is left-orderable iff it is isomorphic to a subgroup of the set of orientation-preserving homeomorphisms on $\mathbb{R}$.
- Used by Mineyev in a proof of the Hanna-Neumann conjecture. Let H and K be non-trivial free groups. Then $\mathrm{rk}(H \cap K) - 1 \leq (\mathrm{rk}(H) - 1)(\mathrm{rk}(K) - 1)$.
- Related to zero divisor conjecture. Let R be a ring without zero divisors and G be torsion free. The conjecture says that the group ring RG has no zero divisors.
  What is known: if G is left-orderable, then RG has no zero divisors.

First example

Def: a group G is *left-orderable* if there exists a total order $\prec$ on the elements of G which is invariant under left-multiplication:

$$g \prec h \iff fg \prec fh, \qquad \forall g, h, f \in G.$$

$(\mathbb{Z}, +)$ has a natural left-order given by

$$\cdots < -1 < 0 < 1 < 2 < \dots$$

The order is clearly invariant under addition.

The definition of left-order is completely symmetric and

$$\cdots < 1 < 0 < -1 < -2 < \dots$$

is another left-order.

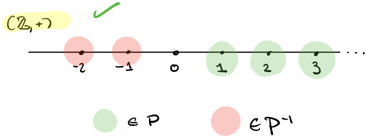
Equivalent notion: positive cone

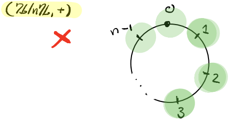
We can look at left-orders in terms of sets called *positive cones*. Roughly speaking, a positive cone is trying to capture the notion of additive positivity in your group.

$P \subset G$ is a *positive cone* for G if

- $PP \subseteq P$ (closed under semigroup operation),
- $G = P \sqcup P^{-1} \sqcup \{1_G\}$ (trichotomy property).

Ex: $(\mathbb{Z}, +)$ admits *a* positive cone. $\mathbb{Z}/n\mathbb{Z}$ does not - no group with torsion has a positive cone.





Suppose a group with torsion has a positive cone. Then is some element g such that $g^n = 1$. Then wlog if $g \in P, g^n = 1 \in P$. Contradiction.

Equivalence of positive cone with left-order

Ex: (continued) $(\mathbb{Z}, +)$ admits this natural left-order

$$\cdots < -1 < 0 < 1 < 2 < \dots$$

The left order is equivalent to defining positive elements

$$P = \{z > 0 \mid z \in \mathbb{Z}\}$$

in the sense that

$$x < y \iff -x + y > 0.$$

In general, if you have a left order, then $P = \{g \in G \mid g \succ 1\}$. Moreover, defining a positive cone also defines an order by $g \prec h \iff g^{-1}h \in P$.

Closure Property 1: subgroups

If $(G, <)$ is a left-orderable group and $H \leq G$, then $(H, <)$ is left-orderable.

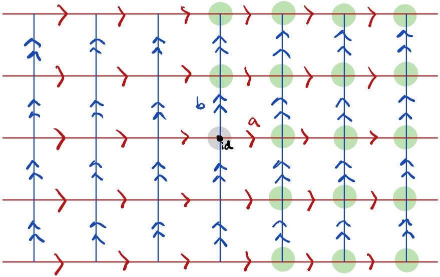
Clearly, if $g < h \iff fg < fh$ for all $g, h, f \in G$, this is also true if $g, h, f \in H$.

ex: $2\mathbb{Z} \leq \mathbb{Z}$.

$$\cdots < -2 < 0 < 2 < 4 < \dots$$

Closure Property 2: extensions

Let's start with an example.
We know that $\mathbb{Z}$ is left-orderable. $\mathbb{Z}^2$ is also left-orderable viewed as $\mathbb{Z} \times \mathbb{Z}$.



A positive cone is for $\mathbb{Z}^2$ is given by

$$P = \{a^n b^m \mid n > 0 \vee n = 0, m > 0\}.$$

This is a lexicographic order with leading factor A, where $A = \langle a \rangle$.
Note that $P = P_A B \cup P_B$, where P_A, P_B are positive cones of $\mathbb{Z}$.

Closure 2: extensions

This idea of composing positive cones generalizes to extensions. Suppose you have $1 \to N \to G \to Q \to 1$, with P_N and P_Q.

Then $P_Q N \cup P_N$ is a positive cone for G since:

Every element can be written as $g = qn$, $q \in Q, n \in N$

Trichotomy: g is such that $q \in P_Q, q \in P_Q^{-1}$ or $q = 1$ and similarly for n, so $G = (P_Q N \cup P_N) \sqcup (P_Q N \cup P_N)^{-1} \sqcup \{1\}$.

Semigroup closure: if $g, h \in P$ such that $g = qn, h = pm$ then $gh = qnpm = qp(p^{-1}np)m = qp(n'm)$ where $n' \in N$. Either q or p are in P_Q, in which case $gh \in P_Q N \subset P$, or $q = p = 1$, in which case $n' = p^{-1}np = n$ and then both $n, m \in P_N$ so $gh \in P_N \subset P$. $\square$

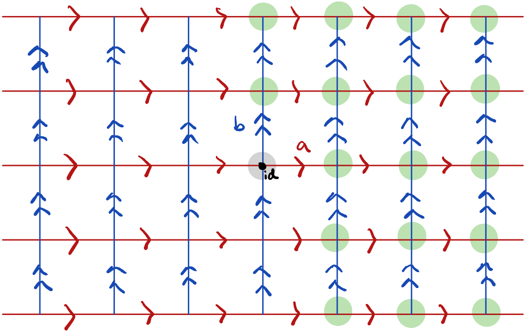
We call this order a *lexicographic order with leading factor Q*.

End of basics. Any questions?

Let’s get to research!

Remark on finite generation

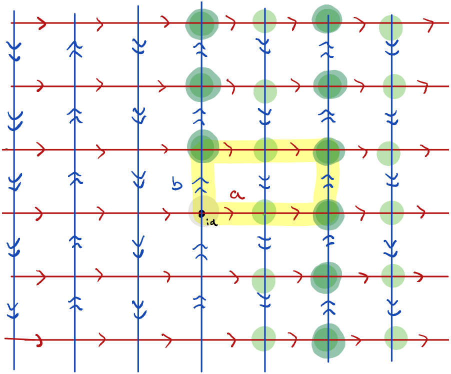
A positive cone for $\mathbb{Z}^2$ is given by two finitely generated positive cones, $P_A = \langle a \rangle^+$, $P_B = \langle b \rangle^+$ such that $P_A B \cup P_B$ is a positive cone for $\mathbb{Z}^2$.



In some sense it's as easy to know if something is positive in $\mathbb{Z}^2$ as it was for $\mathbb{Z}$.
This positive cone for $\mathbb{Z}^2$ is *not* finitely generated (this is just a fact).

Remark on finite generation II

In fact, $\mathbb{Z}^2$ also inherits an isomorphic positive cone as a subgroup of index two in $K_2 = \langle a, b \mid a^{-1}bab = 1\rangle$ with positive cone $\langle a, b\rangle^+$, where $\mathbb{Z}^2 \cong \langle a^2, b\rangle$.



Again, the inherited positive cone for $\mathbb{Z}^2$ is *not* finitely generated.

Remark on finite generation III

Fact: $\mathbb{Z}^2$ has no finitely generated positive cones. This is given by the topological fact that the space of left orders of $\mathbb{Z}^2$, $\mathsf{LO}(\mathbb{Z}^2)$, has no isolated points.

Being finitely generated as a positive cone

- is not closed under taking extensions.
- is not closed under taking finite index subgroups.

Takeaway: being finitely generated is not a very good property to describe how hard it is to determine whether something is in a positive cone.

Research motivation: have a good way to describe positive cone complexity.
We study positive cones and regular languages.

Positive cone language

Let $G = \langle X \mid R \rangle$ be finitely generated.

Let $X^* := \bigcup_{n=0}^{\infty} X^n$.

A *language* is a subset of $L \subset X^*$, and elements of L are called *words*.

We say that L *evaluates* to P if there is an evaluation map $\pi : X^* \to G$ such that $\pi(L) = P$. If P is a positive cone, we call L a *positive cone language*.

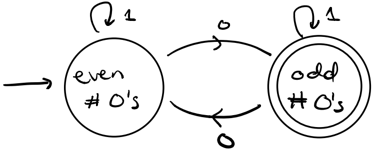
If P has a language L which evaluates to it and L is regular, L is called a *regular positive cone language* and P is called a *regular positive cone*.

Regular languages

Def: A regular language is a language accepted by a *finite state automaton*.

Finite state automata capture the idea of needing *finite memory*.

Ex: This automaton accepts the language of binary strings with an odd number of 0's.

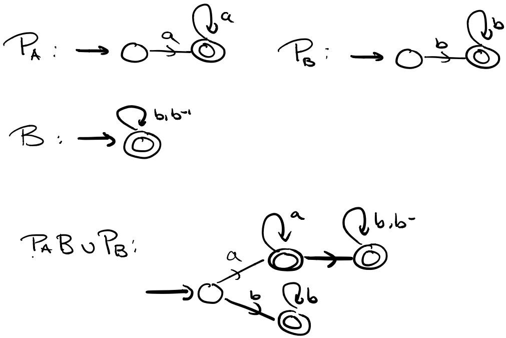


Ex of accepted strings: $10, 000, 01$. Non-examples: $\varepsilon, 1, 00$.

Regular languages are the simplest languages in a classification of languages called the *Chomsky hierarchy*.

Positive cone of $\mathbb{Z}^2$ as a regular language

Ex: $\mathbb{Z}^2 = A \times B$.
$P_A = \langle a \rangle^+$, $B = \langle b \rangle$,
$P_{\mathbb{Z}^2} = P_A B \cup P_B = \{a^n b^m \mid n > 0\} \cup \{b^m \mid m > 0\}$.



Pre-image orders and relation with the word problem

Important: When L represents P, we only need that for every positive element $g \in P$, there exists at least *one* word w of L such that $\pi(w) = g$.

Ex: $(\mathbb{Z}, +) = \langle a \rangle$ admits positive cone language $L = \langle a \rangle^+ := \langle a^n \mid n > 0 \rangle$. $a^2 a^{-1}$ is *not* in L, even though it represents a positive element.

This is very different from the word problem which is about the pre-image language of the identity. Yet, positive cone languages are related to the word problem since if $w \in L$, then $\pi(w) \neq 1$ and therefore $w \notin WP$.

Lemma (Antolín, C. Rivas, and Su, upcoming): If G is such that $\pi^{-1}(P)$ its pre-image positive cone language is regular, then G is trivial.

Overview of research niche

Closure properties of regular positive cones

- Regular positive cones are closed under extensions. (Proof is really easy!)
- Regular positive cones are closed under finite index subgroups. (Su, 2020).
- Having a regular positive cone is independent of generating sets, assuming the sets are finite. (Known fact and easy to proof).
- (*) However, regularity is positive cone dependent. $B(1, q)$ where $q > 1$ has both regular and non-regular positive cones (Antolín, C. Rivas, and Su, upcoming).

Overview of research niche II

Geometric property of regular positive cones

Def: A set is $P \subseteq G$ is *coarsely connected* if it is connected in the Cayley graph up to some R-neighbourhood, for $R \geq 0$.

- (*) Regular positive cone $P \implies$ the set P is coarsely connected (Alonso, Antolin, and C. Rivas, 2020).

Applications:

- Non-abelian free groups have no coarsely connected positive cones and hyperbolic groups with coarsely connected positive cones have to be very distorted in the sense of not being connected by quasi-geodesics (Alonso, Antolin, and C. Rivas, 2020).
- Relatedly, free products have no regular positive cones (Hermiller and Zoran Sunic, 2017) and acylindrically hyperbolic groups have no regular quasi-geodesic positive cones (Su, 2020).

Overview of research niche III

Crossing left-orderable groups with $\mathbb{Z}$:
Something inherent about positive cones change when you cross groups with $\mathbb{Z}$.

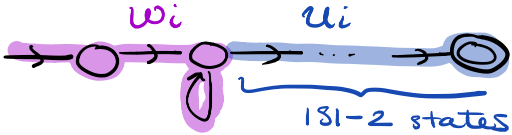
- Let A, B be left-orderable. $A * B$ has no isolated orders (Deroin, Navas, and C. Rivas, 2014). In particular F_2 has no isolated orders.
- $F_2 \times \mathbb{Z}$ has both isolated and non-isolated orders (Mann and Cristobal Rivas, 2018).
- Moreover, $F_2 \times \mathbb{Z}$ has a finitely generated positive cone, which is part of a new infinite family of groups with k-finitely generated positive cones for any $k \geq 3$ (Su, 2020). Whether such a family existed was a question left open by Navas, 2011.
- In general, free products of groups with regular positive cones $A * B$ have a one-counter order (Dicks and Z. Sunic, 2020) but no regular order (Hermiller and Zoran Sunic, 2017).
- (*) $(A * B) \times \mathbb{Z}$ has a regular order (Antolín, C. Rivas, and Su, upcoming).

Geometric interpretation of regularity

Thm (Alonso, Antolin, and C. Rivas, 2020): P is regular $\implies$ P is coarsely connected.

Proof: Let L be a regular language such that $\pi(L) = P$. We will show that there exists an R such that for every $w \in L$, there exists a path $p = p_1 \dots p_n$ from 1 to $\pi(w)$ such that the p_i's are from P to P except for p_1 which starts at 1 and $|p_i| \leq R$.

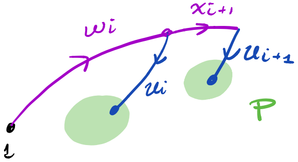
- Let $w = x_1 \dots x_n$, and $w_i = x_1 \dots x_i$. For every prefix w_i, there exists u_i such that $w_i u_i \in L$ and $|u_i| \leq |S|$, the number of states of L.

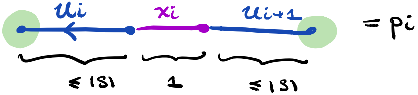


Geometric interpretation of regularity II

Thm (Alonso, Antolin, and C. Rivas, 2020): P is regular $\Longrightarrow$ P is coarsely connected.

- Then $p_i = \pi(u_{i-1}^{-1} x_i u_i)$ is a path of length $\leq 2|S| + 1$ from P to P, (except for p_1 which is 1 to P), and $\pi(p_1 \dots p_n) = \pi(w)$.





- Set $R = 2|S| + 1$.
- $\Longrightarrow$ P is R-connected.

Regularity is positive cone dependent

Thm (Antolín, C. Rivas, and Su, upcoming):
$B(1,q) = \langle a, b \mid aba^{-1} = b^q$ where $q \geq 1$ has both regular and non-regular positive cones.

Non-regular positive cone:

- There is a well-known isomorphism $B(1,q) \cong \mathbb{Z}[1/q] \rtimes \mathbb{Z}$, where a acts by conjugation on b such that $k/q^m \in \mathbb{Z}[1/q]$ corresponds to $a^{-m}b^k a^m$.
- This does match the relation since $aba^{-1} = 1/q^{-1}$ and $b^q = q$.
- Every element $g \in B(1,q)$ can be written in normal form $g = a^n(a^{-m}b^k a^m)$.
  $\implies$ we have a lexicographic order given by $P_{\mathbb{Z}}\mathbb{Z}[1/q] \cup P_{\mathbb{Z}[1/q]}$ as we saw by our extension lemma.

Regularity is positive cone dependent II

$\implies$ we have a lexicographic order given by $P_{\mathbb{Z}}\mathbb{Z}[1/q] \cup P_{\mathbb{Z}}[1/q]$.

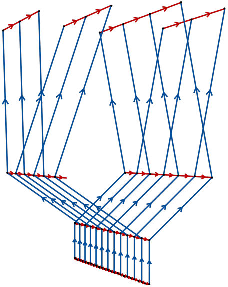


Figure: Cayley graph for $B(1,2)$ where the blue arrows correspond to a. The positive cone is given by everything above the zero level of the tree and part of the zero level. Because of the tree structure, this set is not coarsely connected as branches of the same level are connected by going down the height level, but for every R there exists two branches above zero which are connected by going down $> R$ levels.

Not coarsely connected $\implies$ not regular.

Regularity is positive cone dependent III

Regular positive cone for $B(1, q)$

- $B(1, q)$ has an embedding ρ into $Homeo^+(\mathbb{R})$ given by $\rho(a)(x) = qx$, $\rho(b)(x) = x + 1$ for $x \in \mathbb{R}$.
- We can check that $\rho(aba^{-1})(x) = \rho(b^q)(x) = x + q$.
- Write g in normal form $g = a^n(a^{-m}b^k a^m)$. We observe that $\rho(g)(x) = q^n x + \frac{k}{q^m}$. We can check that this implies that ρ is injective as claimed.
- Let $P_0 := \{g \in B(1, q) \mid \rho(g)(0) > 0\}$. This defines positivity on G except for $Stab(0)$ since $g \in P_0$ acts by translation or dilation. $Stab(0) = \langle a \rangle$, so $P = P_0 \cup \langle a \rangle^+$ defines a positive cone.

This gives us

$$P = \{a^n \mid n > 0\} \cup \{a^n(a^{-m}b^k a^m) \mid k > 0\}$$

which has a regular language representation.

Regularity is positive cone dependent IIII

$$P = \{a^n \mid n > 0\} \cup \{a^n(a^{-m}b^k a^m \mid k > 0)\}$$

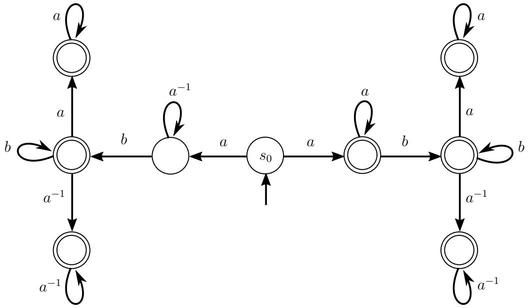
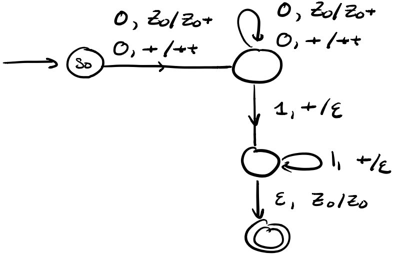


Figure from (Antolín, C. Rivas, and Su, upcoming)

Free products of groups with regular positive cones have a one-counter positive cone

Def: A *one-counter language* is a language which is 'accepted by a finite state automaton equipped with a one-symbol stack'.

Ex: A one-counter automaton accepting the language $\{0^n 1^n \mid n > 0\}$.



It is clear that the set of one-counter languages $\supset$ the set of regular languages since one-counter automata are just regular automata with extra properties.

Free products of groups with regular positive cones have a one-counter positive cone II

Thm (Dicks and Z. Sunic, 2020): Free products of groups with regular positive cones have a positive cone which is accepted by a one-counter language.

Proof: For $G = *_{i=1}^{n} G_i$, write each element in its free product normal form $g = g_1 \dots g_m$ with $g_i \in G_{i_j}$, $i_j \neq i_{j+1}$ and $g_i \neq 1$.
Define a function $\tau : G \to \mathbb{Z}$ where

$$\tau(g) = \#\text{pos syllables}(g) - \#\text{neg syllables}(g) + \#\text{rises}(g) - \#\text{falls}(g)$$

a positive syllable is when $g_i \in P_{i_j}$, a negative syllable is when $g_i \in P_i^{-1}$, a rise is when $g_i g_{i+1}$ is such that $i_j < i_{j+1}$, and a fall is when $i_j > i_{j+1}$.

Free products of groups with regular positive cones have a one-counter positive cone III

$$\tau(g) = \#\text{pos syllables}(g) - \#\text{neg syllables}(g) + \#\text{rises}(g) - \#\text{falls}(g)$$

Properties of τ are as follows

- $\tau(g) = 0 \iff g = 1$ due to normal form.
- τ is odd on $G - \{1\}$ since number of syllables + numbers of rises or falls is odd.
- $|\tau(gh) - (\tau(g) + \tau(h))| \leq 1$ by case checking.

τ defines a positive cone with $P := \{g \in G \mid \tau(g) > 0\}$ because

- For all $g \in G$, $\tau(g) > 0$ or $\tau(g) < 0$ unless $g = 0$.
  $\implies G = P \sqcup P^{-1} \sqcup \{1\}$.
- If $g, h \in P$, then $\tau(gh) = \tau(g) + \tau(h) + \epsilon$, where $\epsilon \in \{-1, 0, 1\}$.
  Since $\tau(g), \tau(h) \geq 1$, we get $\tau(gh) \geq 2 + \epsilon \geq 1$.
  $\implies PP \subset P$.

Example of F_2 being one-counter

The one-counter complexity is given by counting τ. We give this as a transducer.

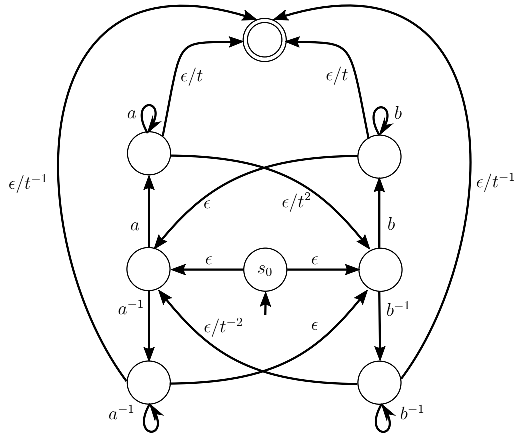


Figure from Antolín, C. Rivas, and Su, upcoming.

Free products of groups with regular positive cones $\times\mathbb{Z}$ have a regular positive cone

Thm (Antolín, C. Rivas, and Su, upcoming) Free products of groups with regular positive cones $\times\mathbb{Z}$ have a regular (and thus coarsely connected) positive cone.
There is a nice group-theoretical interpretation to one-counter.

Intuition: transfer the τ-counter onto the $\mathbb{Z}$-factor.
For $(g, z) \in G \times \mathbb{Z}$, we would like $\tau(g) + z > 0$ to define the positive cone.

Problem: The kernel of τ would contain more than the identity.

Fix: We use that τ is an odd function. Let
$P = \{(g, z) \in G \times \mathbb{Z} \mid \tau'(g, z) = \tau(g) + 2z > 0\}$.

Free products $\times\mathbb{Z}$ have a regular positive cone II

Thm (Antolín, C. Rivas, and Su, upcoming) Free products $\times\mathbb{Z}$ have a regular (and thus coarsely connected) positive cone.

Let $P = \{g \in G \times \mathbb{Z} \mid \tau'(g, z) = \tau(g) + 2z > 0\}$.

At the automaton level:

- We replace x/t^2 by xz^{-1} to 'compensate' for the positivity such that $\tau(g) + 2z = 0$.
- Before accepting a word, we force at least z to be appended at the end, and then freely append by more z's.
- However, we cannot replace x/t by $xz^{-1/2}$, so we multiply each previous state with $+1, -1, 0$ which memorizes the offset value of $\tau'(g, z) = \tau(g) + 2z$ from 0.

Example of regular positive cone for $F_2 \times \mathbb{Z}$

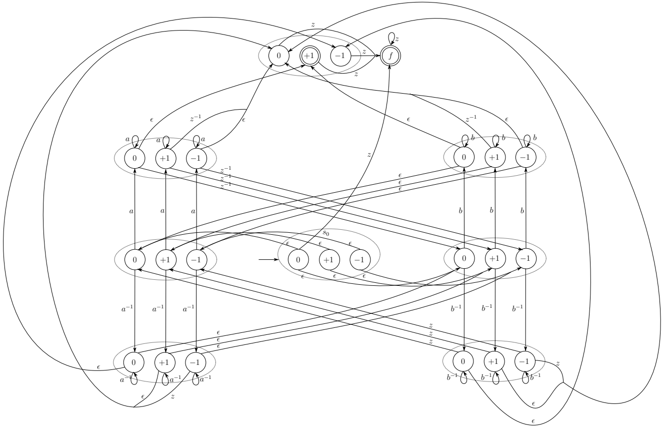


Figure from (Antolín, C. Rivas, and Su, upcoming).

The end!

Advertisement

If you are interested to know why $\mathrm{LO}(\mathbb{Z}^2)$ has no finitely generated positive cones, or in these results of my first paper

1. Regular positive cones are closed under finite index subgroups.
2. Acylindrically hyperbolic groups do not have quasi-geodesic positive cones.
3. $F_2 \times \mathbb{Z}$ has a finitely generated positive cone.
4. The existence of a new infinite family of groups which have k-generated positive cone for any $k \geq 3$.

Go on my website **homeowmorphism.com**.

There's a good quality recording available of my seminar about my first paper with detailed proof sketches.

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