$(\mathrm{Sym}(n), d)$

$$\rightsquigarrow \mathrm{Sym}_{\mathcal{U}} = \prod_{n\in\mathbb{N}} \mathrm{Sym}(n) \,/\, N\mathcal{U}$$

$$N_{\mathcal{U}} := \{ (\sigma_n) \in \prod_{\mathcal{U}} \mathrm{Sym}(n) \mid d(\sigma_n, 1_n) \underset{n\to\mathcal{U}}{\to} 0$$

Γ sofic $\Leftrightarrow$ $\Gamma \hookrightarrow \mathrm{Sym}_{\mathcal{U}}$

Thm Γ amenable $\Rightarrow$ $\Gamma \hookrightarrow \mathrm{Sym}\,\mathcal{U}$.

Thm (Newman-Sohler-Elek)

Γ amenable

$\varphi, \psi : \Gamma \to \mathrm{Sym}_{\mathcal{U}}$ TFAE

1) IRS of φ = IRS of ψ

2) φ and ψ are conjugate

for every finite subset of Γ, $P(F$ fixes rand pt$)$ same for φ and ψ.

Definition: Let $k\in\mathbb{N}$, $w_1, \dots, w_n \in \mathbb{F}_k$.
We say that $w_1, \dots, w_n$ is P-stable (or stable in permutations):

$\iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall m \in \mathbb{N}$

If $\sigma_1, \dots, \sigma_k \in Sym(m)$ satisfy

$d(w(\sigma_1, \dots, \sigma_k), 1) < \delta \quad \forall 1 \le i \le n$

Then $\exists \sigma_1', \dots, \sigma_k' \in Sym(m)$ st

$d(\sigma_i, \sigma_i') < \varepsilon$ and $w_i(\sigma_1', \dots, \sigma_k') = 1$
$\forall 1 \le i \le n$.

<u>Motivating example</u>:

$k = 2, n = 1, w = aba^{-1}b^{-1}$.

Thm (B-L-T)

Let $k \in \mathbb{N}$ and $w_1, \dots, w_n \in \mathbb{F}_k$ & assume $\Gamma := \langle x_1, \dots, x_k \mid w_1, \dots, w_n \rangle$ is <u>amenable.</u>

TFAE:

1) $w_1, \dots, w_n$ is P-stable
2) Every IRS on Γ is a limit of finite index IRS.

<u>ex</u>: $\Gamma = \mathbb{Z}^2$

$Sub(\Gamma)$ = countable and an IRS is just a distribution of atomic measures st the weights add up to 1.

$$\mathbb{Z} \oplus n\mathbb{Z} \xrightarrow[n\to\infty]{} \mathbb{Z} \oplus 0$$

$\rightsquigarrow$ all IRS are limits of fi IRS.

ii) polycyclic groups.

NON-ex: $\exists\, \Gamma$, infinite, amenable, fg and NOT res.-finite

$\rightsquigarrow$ $\exists$ almost no finite index IRS.

In particular, $\delta_{\{1\}}$ is NOT a limit of finite index IRS.

$$\Gamma = \begin{pmatrix} 1 & * & * & * \\ & p^2 & * & * \\ & & p^2 & * \\ 0 & & & 1 \end{pmatrix} \quad * \in \mathbb{Z}[1/p]$$

$$\mathbb{Z} = \begin{pmatrix} 1 & & & \mathbb{Z} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \subseteq \text{Central subgrp}$$

$\rightsquigarrow$ $\Gamma/\mathbb{Z}$ not res. finite.

($\Gamma/\mathbb{Z}$ is non-Hopfian looking at the effect of conjugation by the matrix $\begin{pmatrix} p & & \\ & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ on Γ)

Fact: $\delta_{\mathbb{Z}/6}$ cannot be a limit of finite index IRS of Γ.

2) $\Rightarrow$ 1) "finite index on Γ are dense"

$\Rightarrow w_1, \ldots, w_n$ is P-stable.

Aim prove $\forall \varepsilon > 0, \exists \delta > 0 \ldots$

Assume that is wrong, then $\exists \varepsilon > 0$ $\forall \ell \in \mathbb{N}$:

$w_1, \ldots, w_n$ can be satisfied w/ error $< 1/\ell$ by permutations $\sigma_{1,\ell}, \ldots, \sigma_{k,\ell} \in \mathrm{Sym}(m_\ell)$ and $\nexists$ ε-close solution to $w_1, \ldots, w_n$.

Pick an ultrafilter supported on $\{m_\ell \mid \ell \in \mathbb{N}\}$. For simplicity, let's assume $m_\ell = \ell$.

$S_j := (\sigma_{i,\ell})_{\ell \in \mathbb{N}} \in \mathrm{Sym}\, \mathcal{U} \quad 1 \leq i \leq k$.

Observe that $w_j(S_1, \ldots, S_k) = 1_{\mathrm{Sym}\, \mathcal{U}}$

$\rightsquigarrow \varphi : \Gamma \to \mathrm{Sym}\, \mathcal{U}$.

For $\ell \in \mathbb{N}$, $\sigma_{1,\ell}, \ldots, \sigma_{k,\ell}$ defines an action of F_k on $\{1, \ldots, \ell\}$

$\rightsquigarrow \mu_\ell \in IRS(F_R) \leftarrow$ compact.

$$\rightsquigarrow \mu := \lim_{\ell \to \mathcal{U}} \mu_\ell \in IRS(F_R)$$

If $N := \langle\langle w_1, \dots, w_R \rangle\rangle \trianglelefteq F_R$ then for $w \in N$,

$$P_\mu (w \in H) = 1$$
$$= \int \chi_{[w \in H]} \, d\mu$$

$$\Rightarrow P_\mu (N \subseteq H) = 1$$
$$\Rightarrow \mu \in IRS(\Gamma) = \left\{ \nu \in IRS(F_R) \,\middle|\, \begin{array}{l} \forall H \in \operatorname{supp} \nu \\ N \subseteq H \end{array} \right\}$$

<u>Next step</u>: write μ as the limit
$$\mu = \lim_{\ell \to \mathcal{U}} \nu_\ell \quad w/$$

ν_ℓ finite index IRS of Γ.

With some work, we may assume that ν_ℓ arises from an action of Γ on $\{1, \dots, \ell\}$.

$\rightsquigarrow \varphi : \Gamma \to \mathrm{Sym}_{\mathcal{U}}$ is defined as the ultralimit of these F-actions.

Newman - Sahler Thm $\Rightarrow \varphi$ and ψ are conjugate.

Changing ψ, we may assume $\psi = \varphi$.

Modulo N_n, the representation of φ and ψ agree.

$\rightsquigarrow$ Contradiction to $\exists \varepsilon$ ... $\boxtimes$