Adventures in C^*-algebra, with Jane Panangaden

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Contents

1 May 2-4, 2016

Was a revision of the basics of Statistical Mechanics and basic Quantum Mechanics.

2 May 9, 2016

2.1 Intro to Infinite Dimension

The formulism for the Gibbs state

$$\frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}$$

is not defined for inifinite dimensional spaces. For example, even if the Hamiltonian is a countably infinite dimensional matrix that is 0, the trace is not defined.

We need a new definition of trace. For the trace to make sense, we need the following:

1. $$\sum_{i=1}^{\infty} \langle \psi_i, A\psi_i \rangle$$

   converges.

2. The series is basis independent. Meaning, we need the series to be absolutely convergent. Indeed, if ψ_i, ϕ_i are two ONB's,

$$\sum_i \langle \psi_i, A\phi_i \rangle = \sum_i \sum_j \langle \psi_j, A\phi_i \rangle \langle \phi_i, \psi_j \rangle$$

2.2 Trace Class Operator

Trace Class Operators are basically the operators for which the trace makes sense. Our right now goal is to define $|A|$ for an operator A on a Hilbert space. We remark that for complex numbers,

$$z = |z|e^{i\theta}, \qquad |z| \in [0, \infty), \qquad \theta \in [0, 2\pi)$$
$$z^* z = |z|^2.$$

Now, we define the A^* for an operator A by the property that

$$\langle \psi, A\psi \rangle = \langle A^* \psi, \psi \rangle.$$

Note that for finite dimensional Hilbert spaces, A^* is the conjugate-transpose.

We can define A^*A, which has the property that $(A^*A)^* = A^*A^{**} = A^*A$, so this operator is self-adjoint.

Now, we note that A^*A is a positive operator, for which we can find the square root. Indeed, if A is positive operator, then writing

$$A = \sum_i \lambda_i P_i,$$

we have by definition that $\lambda_i > 0$, so $\sqrt{\lambda_i} > 0$ also, and if

$$B = \sum_i \sqrt{\lambda_i} P_i,$$

then

$$B^2 = \left(\sum_i \sqrt{\lambda_i} P_i \right) \left(\sum_j \sqrt{\lambda_j} P_j \right) = A.$$

The following facts will also be relevant: A *compact operator* is basically are finite-dimensional operator in the following sense:

- Every compact operator is a limit of finite-rank operators.
- Trace class operators are compact.

There will be a spectral theorem for compact operators,

$$A = \sum_{i=1}^{\infty} \lambda_i \langle \psi_i, \cdot \rangle \psi_i,$$

with $\lambda_i \to 0$, which makes sense intuitively.

Definition 1. *We define for* $A \in \mathcal{B}(\mathcal{H}), |A| = \sqrt{A^* A}$*.* A *is* trace class *if* $tr(|A|) < \infty$*.*

If A is trace class, then $\mathrm{tr}(A)$ 'makes sense' because

$$\begin{aligned} \sum_{i=1}^{\infty} |\langle \psi_i, A\psi_i \rangle| &= \sum_{i=1}^{\infty} \left(\langle \psi_i, A\psi_i \rangle \langle A\psi, \psi \rangle \right)^{1/2} \\ &= \sum_{i=1}^{\infty} (\lambda_i \bar{\lambda}_i)^{1/2} \\ &= \sum_{i=1} \langle \psi_i | A | \psi_i \rangle < \infty. \end{aligned}$$

Note:

- A trace class $\Rightarrow$ A compact.
- A trace class $\Rightarrow \{\lambda_i\}$ is in ℓ^1, $\sum_{i=1}^{\infty} |\lambda_i| < \infty$.

3 May 5th 2016

Recap: We wanted to find a formulism, because the Gibbs states does not work for infinite dimension spaces. The idea is to forget $\mathcal{H}$, and work in terms to $\mathcal{B}(\mathcal{H})$, the bounded operators on $\mathcal{H}$.

3.1 Dynamics

Let ψ be our initial state, with Hamiltonian H. The time evolution of the system is given by Schrödinger's equation,

$$-i\frac{\partial}{\partial t}\psi(t) = H\psi(t), \qquad \psi(0) = \psi.$$

This PDE has solution

$$\psi(t) = e^{itH}\psi(0).$$

Consider an ensemble of pure states ψ_i with probability p_i.

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| .$$

We get a new density matrix when we replace ψ_i with $\psi_i(t)$.

$$\rho = \sum_i p_i(e^{-itH}\psi_i, \cdot)e^{itH}\psi_i = e^{-itH}\sum_i p_i(e^{-itH}\psi_i, e^{-itH}e^{itH}\cdot)\psi_i$$

which is an operator $\mathcal{H} \to \mathcal{H}$.

We note the useful fact that, if A is self-adjoint, then e^{itA} is unitary. Indeed,

$$\begin{aligned}(e^{itA})^* &= \left(\sum_{n=0}^{\infty}\frac{itA}{n!}\right)^* \\ &= \frac{(-itA)^n}{n!} \\ &= e^{-itA} = (e^{itA})^{-1}\end{aligned}$$

then, $e^{itH}(e^{-itH})^* = 1$, as expected of unitary operators.

Under this setup, if we have a system in the state ρ, with an observable A, then the expected value is $\mathrm{tr}(\rho A)$. A density matrix ρ induces a linear map

$$\omega_\rho : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$$

$$A \mapsto \mathrm{tr}(\rho A)$$

therefore,

Definition 2. *A* state *is a linear functional on $\mathcal{B}(\mathcal{H})$.*

The time evolution on density matrices is given by

$$\rho(t) = e^{-itH}\rho e^{itH},$$

Using the cyclicity of the trace, we have

$$\mathrm{tr}(e^{-itH}\rho^{itH}A) = \mathrm{tr}(\rho e^{itH}Ae^{-itH})$$

$$A(t) := e^{itH}Ae^{-itH}.$$

This is nice from a mathematical perspective: we can transfer the time evolution to the observable instead of having it act on the state. As we can only physically observe the trace, we do not change the physical theory.

3.2 Structure on $\mathcal{B}(\mathcal{H})$

Definition 3. *Let $\mathcal{L}(\mathcal{H})$ be the space of all linear operators on $\mathcal{H}$. let*

$$\|A\|_{op} = \sup_{\|\psi\|=1\in\mathcal{H}} \|A\psi\|_{\mathcal{H}}$$

where $\|A\psi\|_{\mathcal{H}} = \sqrt{\langle A\psi, A\psi\rangle}$, the L^2-norm on $\mathcal{H}$.

and $\|\psi\|$ is the norm of the vector ψ in H.

Definition 4. *An operator $A \in \mathcal{L}(\mathcal{H})$ is called* bounded *if $\|A\|_{op}\| < \infty$. $\mathcal{B}(\mathcal{H})$ is the space of bounded operators with respect to $\|\cdot\|_{op}$.*

The space of bounded operator is closed under nice algebraic operations. In fact, $\mathcal{B}(H)$ forms a vector space with an algebra:

1. $(A + B)\psi := A\psi + B\psi$
2. $(\lambda A)\psi = \lambda \cdot A\psi$
3. $(AB)\psi = A(B\psi)$

we will also see that there is a nice interplay between the algebraic and the analytic properties of $\mathcal{B}(\mathcal{H})$.

In fact, $\mathcal{B}(\mathcal{H})$ forms a *Banach Algebra*, which means the following:

Definition 5 (Banach Algebra)**.**

- *$\mathcal{B}(\mathcal{H})$ is complete with respect to the $\|\cdot\|_{op}$ norm, as it inherits the completeness of the underlying Hilbert space (exercise)*
- *And $\mathcal{B}(\mathcal{H})$ is closed with respect to the multiplication and satisfies this inequality: $\|AB\|_{op} \leq \|A\|_{op}\|B\|_{op}$.*

(In general, something's a Banach alg if its complete with respect to some norm, closed under mult and satisfies the inequality on the second line).

$\mathcal{B}(\mathcal{H})$ also satisfies the adjoint property:

Proposition 1 (Adjoint property)**.** *For $A \in \mathcal{B}(\mathcal{H})$, there exists a* unique *$A^* \in \mathcal{B}(\mathcal{H})$ such that*

$$\langle A^*\phi, \psi\rangle = \langle \phi, A\psi\rangle, \qquad \forall \phi, \psi \in \mathcal{H}.$$

Another structural property so worthy of note is:

Theorem 1 (Riesz Representation Theorem for Hilbert Space)**.** *For every $\Lambda : \mathcal{H} \to \mathbb{C}$ linear functional, there is a vector $\varphi_\Lambda \in \mathcal{H}$ such that $(\varphi_\Lambda, \psi) = \Lambda(\psi)$.*

This induces an isomorphism between the Hilbert space and its dual.

Proof. For $\varphi \in \mathcal{H}$, define $\Lambda_\varphi(\psi) = (\varphi, \psi)$. This gives you a linear functional for a vector φ. The other direction is more intricate.

Fix $\Lambda \in \mathcal{H}^*$. Then, assuming it's non-trivial, $\dim \operatorname{Im}(\Lambda) = 1$, so, $\dim(\ker \Lambda)^\perp = 1$. Take $z \in \ker(\Lambda)^\perp$ such that $\|z\| = 1$. Since $\Lambda : \mathcal{H} \to \mathbb{C}$, define

$$\varphi_\Lambda := \overline{\Lambda(z)} z.$$

We want to show that $\Lambda(\psi) = (\varphi_\Lambda, \psi)$. Well,

$$(\overline{\Lambda(z)} z, \psi) = \Lambda(z)(z, \psi).$$

We use the fact that $\psi = P_{\ker(\Lambda)}\psi + P_{\ker(\Lambda)^\perp}\psi$. Then,

$$\Lambda(\psi) = \Lambda(P_{\ker(\Lambda)}\psi).$$

and since $z \in \ker(\Lambda)^\perp$ spans it (as it is one-dimensional), we are done. (This needs a bit more work, but like Jane's writing so I'll think about it later. Proof left to the reader of whatever bullshit.) $\square$

We are reader to define the *operator adjoint.* Fix $A \in \mathcal{B}(\mathcal{H})$, and $\varphi \in \mathcal{H}$. Define

$$\Lambda_{A,\varphi}(\psi) = (\varphi, A\psi)$$

Using Riesz, there exists a z such that $\Lambda_{A,\varphi}(\psi) = (z, \psi)$. Define $A^* \in \mathcal{B}(\mathcal{H})$ such that $A^*\varphi = z_{A,\varphi}$. Then,

$$(\varphi, A\psi) = (A^*\varphi, \psi) \qquad \forall \psi, \varphi \in \mathcal{H}.$$

What is A^{**}. Well,

$$(\varphi, A^*\psi) = (A^{**}\varphi, \psi)$$

so $A = A^{**}$, which implies the operator $*$ is an involution.

The next property is the strongest about the structure of $\mathcal{B}(\mathcal{H})$.

3.3 Interaction between $*$ and the norm

The goal is to compute the $\|A^*\|$. Back to Riesz, Given Λ, $z_\Lambda = \overline{\Lambda(z)} z, \qquad \|z\| = 1$.

$$\|z_\Lambda\| = |\overline{\Lambda(z)}| \|z\| = |\Lambda(z)| = \|\Lambda\|_{op}.$$

using the fact that this is all one-dimensional. $\Lambda_{A,\varphi}(\psi) = (\varphi, A\psi)$.

$$\begin{aligned} \|\Lambda_{A,\varphi}\|_{op} = \sup_{\|\psi\|=1} |(\varphi, A\psi)| &\leq \sup_{\|\psi\|=1} \|\varphi\| \|A\psi\| \\ &\leq \sup_{\|\psi\|=1} \|\varphi\| \|A\|_{op} \|\psi\| \\ &= \|\varphi\| \|A\|_{op} \end{aligned}$$

Now,

$$\|A^*\varphi\| = \|z_{\Lambda_{A,\varphi}}\| = \|\Lambda_{A,\varphi}\|_{op} \leq \|\varphi\|_{\mathcal{H}} \|A\|_{op}$$

so

$$\|A^*\|_{op} \leq \|A\|_{op}.$$

And recalling that $A^{**} = A$, we get that

$$\boxed{\|A^*\| = \|A\|.}$$

Also,

$$\|A^*A\| \leq \|A^*\|\|A\| = \|A\|^2.$$

Proposition 2. $\|A^*A\| = \|A\|^2$.

Let $\psi \in \mathcal{H}$. Then

$$\begin{aligned}
\|A\psi\|_{\mathcal{H}}^2 &= |\langle A\psi, A\psi\rangle| = |\langle \psi, A^*A\psi\rangle| \\
&\leq \|\psi\|\|A^*A\psi\| \\
&\leq \|\psi\|^2\|A^*A\|.
\end{aligned}$$

Then, $\|A\|^2 \leq \|A^*A\|$. Then,

$$\boxed{\|A^*A\| = \|A\|^2}.$$

This is called the C^*-condition, and is the most important result of the day.

3.4 Summary

$\mathcal{B}(\mathcal{H})$ is an algebra, a Banach space and an involution which satisfies the properties:

1. $\|AB\| \leq \|A\|\|B\|$.
2. $(AB)^* = B^*A^*$
3. $\|A^*A\| = \|A\|^2$.

This is the definition of being a *C^*-algebra.*

We used $\mathcal{B}(\mathcal{H})$ to introduce them, because: the structure theorem for C^*-algebra tells us that every C^*-algebra is isomorphic to a close subspacce of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

4 May 9, 2016

4.1 Multiplicative Identity

The multiplicative identity might not exist. We want to embed that non-unital C^*-algebra to a unital one, such that the embedding is universal (meaning there's only one way to do it).

Proposition 3. *Let $\mathcal{A}$ be a non-unital C^*-algebra. Then we construct a unital algebra $\tilde{\mathcal{A}}$ such that $\mathcal{A}$ embeds into $\tilde{\mathcal{A}}$.*

Proof. Let
$$\tilde{\mathcal{A}} := \{(\alpha, A), \alpha \in \mathbb{C}, A \in \mathcal{A}\}.$$
Then,
$$(\alpha, A) + (\beta, B) := (\alpha + \beta, A + B),$$
$$(\alpha, A)^* := (\bar{\alpha}, A^*),$$
$$(\alpha, A)(\beta, B) := (\alpha\beta, \alpha B + \beta A + AB),$$
and
$$\|(\alpha, A)\| := \sup_{B \in \mathcal{A}, \|B\|=1} \|\alpha B + AB\|_{\mathcal{A}}.$$
Notice that the multiplication has some 'twist' such that we can define $(1, 0)$ to be the unital element of this new algebra $\tilde{\mathcal{A}}$, and that $(0, A)$ acts in $\tilde{\mathcal{A}}$ as A does on $\mathcal{A}$, which proves that we have an embedding of $\mathcal{A}$ into something unital. □

Proposition 4. *$\tilde{A}$ is a C^*-algebra.*

Proof. We won't go through all the criteria, but let's check that $\|\cdot\|$ is a norm.

1. Triangle inequality

$$\begin{aligned}\|(\alpha, A) + (\gamma, C)\|_{\tilde{\mathcal{A}}} &= \|(\alpha + \gamma, A + C)\|_{\tilde{\mathcal{A}}} \\ &= \sup_{\|B\|=1} \|(\alpha + \gamma)B + (A + C)B\|_{\mathcal{A}} \\ &\leq \sup_{\|B\|=1} \|\alpha B + AB\|_{\mathcal{A}} + \sup_{\|B\|=1} \|\gamma B + CB\|_{\mathcal{A}} \\ &= \|(\alpha, A)\|_{\tilde{\mathcal{A}}} + \|(\gamma, C)\|_{\tilde{\mathcal{A}}}.\end{aligned}$$

2. Positive definite:

Now suppose that $\|(\alpha, A)\| = 0$. Case 1: $\alpha = 0$. Then,
$$\|(0, A)\|_{\tilde{\mathcal{A}}} = \sup_{\|B\|=1} \|AB\| \leq \|A\|.$$
This is actually attained: if $B = \frac{A^*}{\|A^*\|} = \frac{A^*}{\|A\|}$.
$$\frac{\|AA^*\|}{\|A\|} = \|A\|.$$
also if $\|(0, A)\| = \|A\| = 0$, then $A = 0$.

Case 2: $\alpha \neq 0$. W.l.o.g. take $\alpha = 1$,
$$\|B - AB\| \leq \|B\|.$$
$$\begin{aligned}\|(1, -A)\|_{\mathcal{A}} &= \sup_{\|B\|=1} \|B - AB\| \\ &\geq \frac{1}{\|B\|}\|B - AB\| \qquad \forall B \in \mathcal{A} \\ \|B - AB\|_{\mathcal{A}} &\leq \|B\|\|(1, -A)\|_{\tilde{\mathcal{A}}}\end{aligned}$$

if $\|(1,-A)\| = 0$, then $B = AB$, by the thing above, $B = AB, \quad \forall B \in \mathcal{A}$, (starring both sides) $B = BA^*, \quad \forall B \in \mathcal{A}$, $B = A^*$, $A = AA^*$, and also $A^* = AA^*$, so, $A = A^*$.

So then we get $B = AB = BA$, so A is the identity for all $A \in \mathcal{A}$, but we assumed that $\mathcal{A}$ wasn't unital.

3. C^* condition.

$$\begin{aligned}
\|(\alpha, A)\|^2 &= \sup_{\|B\|=1} \|\alpha B + AB\|^2 \\
&= \sup_{\|B\|=1} \|(\bar{\alpha} B^* + B^* A^*)(\alpha B + AB)\| \\
&= \sup_{\|B\|=1} \|\bar{\alpha}\alpha B^* B + \alpha B^* A^* B + \bar{\alpha} B^* AB + B^* A^* AB\| \\
&= \sup_{\|B\|=1} \|B^*(\bar{\alpha}\alpha B + \alpha A^* B + \bar{\alpha} AB + A^* AB)\| \\
&= \sup_{\|B\|=1} \|B^*(\bar{\alpha}\alpha + \alpha A^* + \bar{\alpha} A + A^* A)B\| \\
&\leq \|(\alpha, A)^*(\alpha, A)\| \\
&\leq \|(\alpha, A)^*\|\|(\alpha, A)\|
\end{aligned}$$

so we have $\|(\alpha, A)\| \leq \|(\alpha, A)^*\|$, and by interchanging the roles of (α, A) and $(\alpha, A)^*$, we get that

$$\|(\alpha, A)^*(\alpha, A)\| \leq \|(\alpha, A)^*\|\|(\alpha, A)\| \leq \|(\alpha, A)\|^2.$$

4. $\|1\| = 1$. Since $\|1\| = \|11^*\| = \|1\|^2$, either $\|1\| = 0$ or $\|1\| = 1$. Now, notice that for all $A \in \mathcal{A}$,

$$\|A\| \leq \|1\|\|A\|,$$

so, if $\|1\| = 0$, $\|A\| = 0$ for all A, so $\mathcal{A}$ is trivial. Unless $\mathcal{A}$ is trivial, $\|1\| = 1$. □

4.2 Quotients of C^*-algebras

Definition 6. *If $\mathcal{A}$ is a C^*-algebra, $\mathcal{I} \subset \mathcal{A}$ is a left (right) ideal, if $B \in \mathcal{A}, A \in \mathcal{I}, BA \in \mathcal{I}, AB \in \mathcal{I}$, also $\mathcal{I}$ is closed under addition.*

Proposition 5. *If $\mathcal{I}$ is a left ideal closed under the $*$ operation, then it is also a right ideal.*

Proof. $A \in \mathcal{I}$, $B \in \mathcal{I} \Rightarrow AB \in \mathcal{I}$ (left ideal). Now, $B^* \in \mathcal{I}$ (by assumption) $\Rightarrow A^*B^* \in \mathcal{I}$. Again, $(A^*B^*)^* \in \mathcal{I}$, but $(A^*B^*)^* = BA \in \mathcal{I}$. □

Definition 7. *A* quotient space *is defined as $\mathcal{A}/\mathcal{I}$, where $\mathcal{I}$ is an ideal. Therefore, every element of $\mathcal{A}/\mathcal{I}$, are equivalence classes up to an operation with an element of $\mathcal{I}$. More formally,*

$$\mathcal{A}/\mathcal{I} := \{[A] : A \in \mathcal{I}\},$$

and

$$[A] := \{A + I : I \in \mathcal{I}\},$$

$$[A] \times [B] := [A \times B].$$

$$\|[A]\| := \inf\{\|A + I\| : I \in \mathcal{I}\}.$$

4.3 Spectral Theory

4.3.1 Finite dimensional version

$\mathcal{H} : \dim \mathcal{H} < \infty$. Let our space be $\mathcal{B}(\mathcal{H})$.

Theorem 2 (Spectral Theorem). *A is normal ($A^*A = AA^*$). Then,*

$$A = \sum_{i=1}^{n} \lambda_i P_i.$$

where λ_i are the eigenvalues of A, and P_i are the projections onto the eigenspaces E_{λ_i}.

Remarks:

- We call $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ *the spectrum* of A, and attached to $\sigma(A)$ are probabilities distributions on $\sigma(A)$.

- Fix a density matrix ρ, then
$$p_i = \mathrm{tr}(\rho P_i).$$
where P_i is the projection operator associated with λ_i, and corresponds to the probability of observing λ_i given an observable A and a state ρ.

- We can take functions of operators,
$$p(x) = \sum_{i=1} c_i x^i,$$
$$p(A) := \sum_{i=1} c_i A^i,$$
and under the spectral decomposition,
$$p(A) = \sum_{i=1} p(\lambda_i) P_i.$$
since the polynomials are dense in the space of continuous functions, we may define $f(A)$ for a continuous function, as a limit of polynmials, and get a spectral decomposition
$$f(A) := \sum_{i} f(\lambda_i) P_i.$$

- The connection between the analytic and algebraic parts of $\mathcal{B}(\mathcal{H})$ is the following:
$$\|A\|_{op} = \sup_{\|\psi\|=1} \|A\psi\| = \lambda_{\max}$$

- We may characterize self-adjoint and unitary operators by their eigenvalues: self-adjoint operators have real eigenvalues, and unitary operators have eigenvalues on the unit circle in $\mathbb{C}$.

4.3.2 The infinite-dimensional setting

The plan:

1. Define $\sigma(A), A \in \mathcal{A}$. It has a much richer structure.
2. Connection with $\|\cdot\|_{\text{op}}$. $\sigma(A)$ is a bounded set, so we can define its radius
$$R(\sigma(A)) = \|A\|_{\text{op}}.$$
3. Functional Calculus version of the Spectral Theorem. This will give us a way to define $f(A)$ with continuous functions f. (This is hard to do, but very useful).
4. Spectral Measure. If anyone knows the Other Riesz Representation Theorem, you know that you can take a linear functional Λ, and associate a measure μ_Λ, such that μ_Λ replaces
$$\Lambda(f) = (\psi, f(A)\psi).$$
(See Wikipedia article on the (Other) Riesz Representation Theorem, also known as the Riesz-Markov one.)

Let's execute the plan.

4.4 Step 1

In the finite setting, if $\lambda \in \sigma(A)$ then, $\exists v \neq 0 \in \mathcal{H}$ such that $Av = \lambda v$, if and only $\ker(A - \lambda I)$ is non-trivial. Then, this implies $(A - \lambda I)^{-1}$ does not exists.

Definition 8. *Let $\mathcal{A}$ be a C^*-algebra. For $A \in \mathcal{A}$,*
$$\sigma(A) := \{\lambda : (A - \lambda I)^{-1} \text{ does not exists } \}.$$

Now, keep in mind that, for infinite-dimensional spaces, this set looks a lot of complicated than in the finite-dimensional case. This is because an injective operator might not be surjective, or even have a dense range in infinite dimensions, or have a bounded inverse, so you can get things that don't look like eigenvalues at all.

5 May 11, 2016

5.1 Spectral Radius

Last time, we defined the spectrum of A to be
$$\sigma(A) := \{\lambda : \lambda I - A \text{ is not invertible } \}.$$

In the finite-dimensional case, if$\mathcal{A} = \mathcal{B}(\mathcal{H})$, then $\sigma(A) = \{\text{eigenvalues of} A\}$, because we can use the rank-nullity theorem.

If $\dim \mathcal{H} = \infty$, $(\lambda I - A)$ not invertible can mean:

1. May not be injective $\Rightarrow \lambda$ is an eigenvalue (this is the only case that's analogous to the finite dimensional one).

2. May be injective, but not surjective.

3. May have a set theoretic inverse, but it may not be bounded.

Recall: if $\dim \mathcal{H} < \infty$, $\|A\|_{\text{op}} = \lambda_{\max}$. i.e. the biggest eigenvalue.

Definition 9. *For $A \in \mathcal{A}$, the* spectral radius *of A, $R(A)$ is*

$$R(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

This matches the biggest eigenvalue in the case that A is normal.

Lemma 1. $R(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.

Proof. (Upper bound): Want to show that $\liminf \|A^n\|^{1/n} \geq R(A)$. Fix λ such that $|\lambda| > \|A^n\|^{1/n}$ for some n. Then

$$\left\| \left(\frac{A}{\lambda}\right)^n \right\| < 1.$$

Now we consider the series

$$\lambda^{-1} \sum_{m\geq 0} \left(\frac{A}{\lambda}\right)^m,$$

We will show that this series converges, by showing that the sequence of partial sums is Cauchy. and using the completeness of $\mathcal{B}(\mathcal{H})$. Well,

$$\left\| \sum_{M_1 \leq m \leq M_2} \left(\frac{A}{\lambda}\right)^m \right\| \leq \sum_{M_1 \leq m \leq M_2} \left\| \left(\frac{A}{\lambda}\right)^n \right\|^{p_m} \left\| \frac{A}{\lambda} \right\|^{q_m}.$$

using the Euclidean division algorithm, for each m we have $m = np_m + q_m, \quad 0 \leq q_m < n$. Substracting two geometric series,

$$= \sum_{0\leq q<n} \left\| \frac{A}{\lambda} \right\|^q \left\{ \frac{\left\| \left(\frac{A}{\lambda}\right)^n \right\| \left[(1 - \left\| \left(\frac{A}{\lambda}\right)^n \right\|^{p_2}) - (1 - \left\| \left(\frac{A}{\lambda}\right)^n \right\|^{p_1}) \right]}{1 - \left\| \left(\frac{A}{\lambda}\right)^n \right\|} \right\}$$
$$\leq C \frac{(\|\frac{A}{\lambda})^n \|}{1 - (\|\frac{A}{\lambda})^n \|} \cdots$$

so this is Cauchy. Now, we assumed that $|\lambda| > \|A^n\|^{1/n}$ for some n, so $\lambda \notin \sigma(A)$. Then, $R(A) \leq \|A^n\|^{1/n}$ for all n, so $R(A) \leq \liminf \|A^n\|^{1/n}$. $\square$

The proof of the lower bound is more complicated. I was troubleshooting the above equation, so. Missed it! The point was to show that it's painful to develop these spectral tools, but worth it: we will be proving cool things in the sequel.

5.2 Applications of the Spectral Radius

Using the formula
$$R(A) = \lim \|A^n\|^{1/n} \leq \lim \|A\|^{(n \cdot 1/n)} = \|A\|.$$
So then, for any $A \in \mathcal{A}$, $R(A) \leq \|A\|$.

Proposition 6. *If A is normal ($A^*A = A^*A$), then $R(A) = \|A\|$.*

Remark: Normality is also the condition of an operator for the spectral theorem to apply in finite dimensions.

Proof.

$$\begin{aligned}
\|A^{2^n}\|^2 &= \|(A^{2^n})^*(A^{2^n})\| \\
&= \|(A^{2^n})(A^{2^n})^*\| && \text{by normality} \\
&= \|(A^*A)^{2^n}\| \\
&= \|\{A^*A)^{2^{n-1}}\}^2\| \\
&= \|\{A^*A)^{2^{n-1}}\}\|^2 && \text{C-starness} \\
&= \|\{A^*A)\}\|^{2^{n+1}}
\end{aligned}$$

Therefore, $R(A) = \lim \|A^{2^n}\|^{2^{-n}} = \lim \|A\|^{2^n 2^{-n}} = \|A\|$. □

Proposition 7 (!)**.** *Let $\mathcal{A}$ be a $*$-algebra. If there is a norm $\|\cdot\|$ making $\mathcal{A}$ a C^*-algebra, then it is the unique such norm.*

This says that the C^*-algebra structure is actually uniquely determined by the algebraic structure, because there can only really be one norm!

Proof. Suppose A is normal. Then $R(A) = \|A\|$. Its norm is fixed by the algebraic structure. If $A \in \mathcal{A}$, consider A^*A, which is normal. Then $\|A\| = \sqrt{\|A^*A\|} = \sqrt{R(A^*A)}$. □

Note, here we used the definition of $R(A)$ as the $\sup\{|\lambda| : (\lambda I - A)^{-1}$ does not exists $\}$, so the value is guaranteed to be a fixed real number which depends on what is invertible (algebraic structure).

Proposition 8. *Suppose $\mathcal{A}, \mathcal{B}$ are C^*-algebras. Let $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-morphism. Then $\|\pi(A)\|_{\mathcal{B}} \leq \|A\|_{\mathcal{A}}$.*

Proof. Let A be normal. If $\lambda \notin \sigma(A)$, then $(\lambda I - A)^{-1}$ exists, so

$$\pi((\lambda I - A)^{-1}) = (\lambda \pi(I) - \pi(A))^{-1} = (\lambda I_{\mathcal{B}} - \pi(A))^{-1}$$

so, $\lambda \notin \sigma(\pi(A))$. $\Rightarrow \sigma(\pi(A)) \subset \sigma(A)$. $\Rightarrow R(\pi(A)) \leq R(A)$, $\Rightarrow \|\pi(A)\| \leq \|A\|$. For A general, A^*A is normal.

$$\|\pi(A)\|^2 = \|\pi(A)^*\pi(A)\| = \|\pi(A^*A)\| \leq \|A^*A\| = \|A\|^2.$$

□

6 May 12, 2016

Next week: M: 2-4, Tu: 3:30-5:30, W: 3:30-5:30. No class following Monday.

Last time: we proved a formula for the spectral radius,

$$R(A) = \lim_{n\to\infty} \|A^n\|^{1/n}.$$

In particular, this means we can bound the spectral radius above by

$$R(A) \leq \|A\|,$$

with equality if A is normal (which if the finite dimensional requirement for the spectral theorem). This had various consequences:

1. C^* algebra norms are unique.
2. Maps between C^* algebras which preserve the algebraic structure also preserve the analytic structure.

Definition 10. *U is* unitary *if $U^*U = UU^* = I$, A is* self-adjoint *if $A = A^*$.*

Then, for $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ $\langle Uv, Uv, = \langle v, v\rangle$, and $\langle v, Av\rangle = \langle Av, v\rangle$. So if $Uv = \lambda v$, then $|\lambda| = 1$. and if $Av = \lambda v$, then $\bar{\lambda} = \lambda$, meaning $\lambda \in \mathbb{R}$. This is why we expect observables to be self-adjoint, because we make real observations.

Proposition 9. *If $U \in \mathcal{A}$ is unitary, then $\sigma(U) = S^1$, the unit circle in $\mathbb{C}$. If A is self-adjoint, $\sigma(A) \subset [-\|A\|.\|A\|]$.*

Note: we assume everything has a unit, because of the embedding.

Proof. (unitary) Since U is normal,

$$R(U) = \|U\| = 1,$$

since

$$\|U^*U\| = \|U\|^2 = \|1\| = 1.$$

$\square$

Suppose $\lambda \in \sigma(U)$. Then $\bar{\lambda} \in \sigma(U^*) = \sigma(U^{-1})$. $\bar{\lambda}^{-1} \in \sigma(U)$, then $|\lambda^{-1}| \leq 1$, $|\lambda| \leq 1 \Rightarrow |\lambda| = 1$. This is because $(\lambda I - U)^{-1}$ exists $\iff$ $(\bar{\lambda} I - U^*)^{-1} = ((\lambda I - U)^{-1})^*$.

(Self-Adjoint) Fix λ such that $\left|\frac{1}{\lambda}\right| \geq \|A\| = R(A)$. $\Rightarrow \frac{1}{\lambda} \notin \sigma(A)$ and $\frac{i}{|\lambda|} \notin \sigma(A)$. $\Rightarrow$ the inverse $I + i|\lambda|A$ exists. Define

$$B := (I - i|\lambda|A)(I + i|\lambda|A)^{-1}.$$

One can check that $B^*B = I$, so $\sigma(B) \subset S^1$. Now,

$$\frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha} \in \sigma(B) \iff \left(\frac{1 - i|\lambda|\alpha}{1 + i|\lambda|\alpha} I - B\right) \text{ if not invertible}$$

$$\Rightarrow \left|\frac{1-i|\lambda|\alpha}{1+i|\lambda|\alpha}\right| = 1 \iff \alpha \in \mathbb{R}.$$

Now,

$$\left(\frac{1-i|\lambda|\alpha}{1+i|\lambda|\alpha} - B\right) = 2i|\lambda|(I+i|\lambda|\alpha)^{-1}(A-\alpha I)(I+i|\lambda|A)^{-1}$$

if $A-\alpha I$ is invertible, then $\alpha \in \mathbb{R}$.

6.1 Functional Calculus

For $\mathcal{A} \in \mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} < \infty$. Then, if A is normal, use its spectral decomposition

$$A = \sum_i \lambda_i P_i$$

, and we can define for $p(x) = \sum_i a_i x^i$,

$$p(A) := \sum_i a_i A^i = \sum_i a_i \left(\sum_j \lambda_j P_j\right)^i = \sum_j p(\lambda_j) P_j$$

Using Stone-Weierstrass, we can approximate continuous functions uniformly by polynomials. Using $\|f\|_\infty = \sup |f(x)|$, if we have f continuous, $p_n \to f$ with respect to $\|\cdot\|_\infty$. Now,

$$\begin{aligned}\|p_n(A) - p_m(A)\| &= \|\sum_i (a_{n,i} - a_{m,i} A^i\|_{\text{op}} \\ &= |\sum_i (a_{n,i} - a_{m,i})(\lambda)^i| \le \|p_n - p_m\|_\infty\end{aligned}$$

where λ is some eigenvalue of A. Then, $p_n(A)$ converges in $\|\cdot\|_{\text{op}}$ by completeness of the C^* algebra, so we can define

$$f(A) := \lim p_n(A).$$

6.2 Spectral Theorem

Theorem 3 (Spectral Theorem)**.** *Let $A \in \mathcal{A}$ be normal. Then there is a map*

$$\Phi_A : C(\sigma(A)) \to \mathcal{C}^*(A)$$

where $C(\sigma(A))$ is the set of continuous functions on the spectrum of A, and $\mathcal{C}^*(A)$ is the subalgebra generated by A and I, such that

1. *Φ_A is an isomorphism*
2. *Φ_A is an isometry with $\|\cdot\|_\infty$ on $C(\sigma(A))$.*
3. *$\Phi_A(1) = I$, where $1(x) = 1$.*

4. $\Phi_A(id) = A$, $id(x) = x$.

Note: $\Phi_A(f) := f(A)$.

Theorem 4 (Gelfand rep theorem)**.** *Every unital, commutative C^* algebra is isomorphic to $C(X)$ for X a compact, Hausdorff topological space.*

Proof.

Definition 11. *Let $\mathcal{A}$ be a C^* algebra. A* character *on A is a surjective homomorphism $\gamma : \mathcal{A} \to \mathbb{C}$.*

Let $\hat{\mathcal{A}}$ be the set of all characters on $\mathcal{A}$, $\mathcal{A} \subset \mathcal{A}^*$, the set of linear functionals on $\mathcal{A}$. The goal is to send $\mathcal{A}$ to $C(\hat{\mathcal{A}})$. But with respect to what topology? How are we continuous on $\hat{A}$?

Answer: the weak-$*$ topology.

Let X be a Banach space. Let X^* be the set of linear functionals on X. We want to embed $X \hookrightarrow X^{**}$. We are going to do that by using the evaluation map ι. Let $A \in X$, $\Lambda \in X^*$. Then

$$\iota(A)(\Lambda) := \Lambda(A).$$

The weak-$*$ toppology is the weakest topology on X^* which makes everything in $\iota(X) \subset X^{**}$ continuous.

Proposition 10. *The weak-$*$ topology is the topology of pointwise convergence. A sequence $\{\Lambda_n\} \subset X^{**}$ converges $*$-weakly $\iff$ $\{\Lambda_n(A)\}$ converges in $\mathbb{C}$ $\quad \forall A \in X$.*

Proof. Equip $\hat{\mathcal{A}}$ with the subspace topology induced by the weak-$*$ topology on $\mathcal{A}^*$. $\square$

So anyway, we have $\hat{\mathcal{A}} \subset \mathcal{A}^*$ with weak-$*$-topology.

Proposition 11. *$\mathcal{A}$ is Hausdorff.*

Proof. Suppose $\Lambda \neq \Lambda' \in \hat{\mathcal{A}}$. Then $\exists A \in \mathcal{A}$ such that $\Lambda(A) \neq \Lambda'(A)$. Then, separate $\Lambda(A), \Lambda'(A)$ by U, U' in $\mathbb{C}$. Define:

$$V = \{f \in \hat{\mathcal{A}} : f(A) \in U\}$$
$$V' = \{f \in \hat{\mathcal{A}} : f(A) \in U'\}$$

$\square$

$\hat{\mathcal{A}}$ is compact is much more involved. First we need a result which relates $\sigma(A)$ to the set $\hat{\mathcal{A}}$.

Lemma 2. *There is a bijection between $\hat{\mathcal{A}}$ and the set of maximal ideals in $\mathcal{A}, M(\mathcal{A})$.*

Proof.

$$\gamma \mapsto \ker(\gamma) \in M(\mathcal{A})$$

$$\pi : \mathcal{A} \to \mathcal{A}/I \hookleftarrow I$$

Suppose I is maximal. Then $\mathcal{A}/I$ has no proper ideal. Let $A \in \mathcal{A}/I$, such that $A \neq 0$. If A is not invertible, $\langle A \rangle$ is a proper ideal, so A must be invertible.

We know that $\sigma(A)$ is non-empty, from proof that $R(A) = \lim \|A^n\|^{1/n}$. Choose $\lambda \in \sigma(A)$. $\lambda I - A$ is not invertible implies that $A = \lambda I$.

$$\Rightarrow \mathcal{A}/I \cong \mathbb{C}$$

$\Rightarrow \pi : \mathcal{A} \to \mathcal{A}/I \cong \mathbb{C}$ is a character.

Conversely, let γ be a character. $\gamma : \mathcal{A} \to \mathbb{C}$,

$$\Rightarrow \mathcal{A}/\ker\gamma \cong \mathbb{C}.$$

Since $\mathbb{C}$ is a field, $\ker\gamma$ must be maximal (it's a fact from algebra). $\square$

Lemma 3. $\sigma(A) = \{\gamma(A) : \gamma \in \hat{\mathcal{A}}\}$.

Proof. Fix $A \in \mathcal{A}$. Choose $\lambda \in \sigma(A) \Rightarrow \lambda I - A$ is not invertible $\Rightarrow \langle(\lambda I - A)\rangle$ is proper, since $1 \notin \langle\lambda I - A\rangle$. Consider the family of all ideals containing A but not 1, ordered by inclusion.

Zorn's lemma says that there is a proper maximal ideal containing A, call is $I_{\max}$. From the previous, there exists $\gamma \in \hat{\mathcal{A}}$ such that $\ker\gamma = I_{\max} \Rightarrow \lambda I - A \in \ker\gamma$, so $\gamma(\lambda I - A) = 0 \Rightarrow \gamma(A) = \lambda$.

Suppose $\exists\gamma \in \hat{\mathcal{A}}$ such that $\gamma(A) = \lambda$. Then,

$$\gamma(\lambda I - A) = 0$$

$$\lambda I - A \in \ker\gamma$$

$\Rightarrow \lambda I - A$ is a proper maximal ideal of $\mathcal{A}$ by previous lemma. $\Rightarrow \lambda I - A$ is not invertible. $\square$

Proposition 12. *$\hat{\mathcal{A}}$ is weak-$*$ compact.*

Proof. $\gamma \in \hat{\mathcal{A}}$, from the thing we've just proved, $|\gamma(A)| \leq R(A) \leq \|A\|$.

$$\Rightarrow \|\gamma\|_{\mathrm{op},\mathcal{A}^*} \leq 1.$$

$\Rightarrow \hat{\mathcal{A}}$ is contained in the unit ball of $\mathcal{A}^*$ (with $\|\cdot\|_{\mathrm{op},\mathcal{A}^*}$)

Theorem 5 (Banach-Algoglu Theorem)**.** *The unit ball is weak-$*$ compact.*

$\hat{\mathcal{A}}$ is contained in a compact set $\Rightarrow$ we just need to show its closed. $\square$

Definition 12. *The Gelfand transform is the map*

$$\Gamma : \mathcal{A} \to C(\hat{\mathcal{A}})$$

$$A \mapsto \hat{A}.$$

where

$$\hat{A} : \hat{\mathcal{A}} \to \mathbb{C},$$

$$\gamma \mapsto \gamma(A)$$

Note: $\hat{A}$ is 'evaluation at A' and hence weak-$*$ continuous. By lemma 3, $\Gamma(A)(\hat{\mathcal{A}}) = \sigma(A)$.

We need check these:

- Γ is a homomorphism.
- Γ separates points, in the sense that if $\gamma_1 \neq \gamma_2 \in \hat{\mathcal{A}}$, there exists $A \in \mathcal{A}$ such that $\gamma_1(A) \neq \gamma_2(A)$.
- $\Gamma(\mathcal{A})$ contains contant functions on $\hat{\mathcal{A}}$.
- $\Gamma(\mathcal{A})$ is self-adjoint (closed under $*$).

We'll just prove the last statement here: Let $A \in \mathcal{A}$ self-adjoint. Then for all $\gamma \in \hat{\mathcal{A}}$,

$$\underbrace{\Gamma(A)(\gamma)}_{\in \mathbb{C}} = \hat{A}(\gamma) = \gamma(A) \in \sigma(A) \subset \mathbb{R}$$

Then, $\Gamma : \hat{\mathcal{A}} \to \mathbb{C}$, so $\overline{\Gamma(A)} = \Gamma(A)$. If A is general, $A = B + iC$, where B, C are self-adjoint is a fact of Lyfe. So, $\Gamma(A^*) = \Gamma(B - iC) = \Gamma(B) - i\Gamma(C) = \overline{\Gamma(A)}$.

Theorem 6 (Stone Weierstress (Awesome Version))**.** *Let X be a compact Hausdorff space, $\mathcal{B}$ a subalgebra of $C(X)$ which*

1. *separates points*
2. *contains the constant functions*
3. *is self-adjoint.*

then $\mathcal{B}$ is uniformly dense in $C(X)$, (i.e. with respect to $\|\cdot\|_\infty$).

Note: $\Gamma(\mathcal{A})$ satisfies all those properties, so $\Gamma(\mathcal{A})$ is uniformely dense in $C(\hat{\mathcal{A}})$. Now we are ready to prove Gelfand!

Proof. (Gelfand) $\mathcal{A}$ is commutative $\Rightarrow$ every element is normal ($AA^* = A^*A$). By lemma 3, $\|\Gamma(A)\|_\infty = R(A) = \|A\|$, so Γ is an isometry ($\Rightarrow \Gamma$ is injective).

By Stone-Weierstrass, the range of Γ is dense. Dense range and isometry gives that Γ is surjective.

Indeed, if $\Gamma : X \to Y$ is an isometry, and $\Gamma(X)$ is dense in Y, then let $y \in Y$. There is a sequence $\{x_n\} \subset X$ such that $\Gamma(x_n) \to y$ (density). $\{\Gamma(x_n)\}$ is Cauchy in Y, so $\{x_n\}$ is Cauchy in X, and letting $x_n \to x$, $\lim \Gamma(x_n) = \Gamma(x) = y$. □

7 May 16, 2016

Last time: Structure theorem for commutative unital C^* algebra. Let $\mathcal{A}$ be a C^* algebra, $\hat{\mathcal{A}}$ the set of characters on $\mathcal{A}$ (surjective homomorphisms).

$$\Gamma : \mathcal{A} \to C(\hat{\mathcal{A}})$$

$$A \to \hat{A}.$$
$$\hat{A} : \gamma \mapsto \gamma(A).$$

where Γ is the Gelfand transform. It is an isometric isomorphism ($\|\cdot\|_\infty$ on $C(\hat{\mathcal{A}})$).

We had the lemma $\sigma(A) = \{\gamma(A) : \gamma \in \mathcal{A}\}$.

Theorem 7. *Let $A \in \mathcal{A}$ be normal. Let $C^*(A)$ be the smallest C^*-algebra containing A, i.e. it contains all polynomials in $(A, A^*, 1)$. Then there exists $\Phi_A : C(\sigma(A)) \to C^*(A)$ such that*

1. *Φ_A is an isometric isomorphism*
2. *$\Phi_A(id) = A$*
3. *$\Phi_A(1) = 1$*

in that sense, we may interpret $\Phi_A(f) = f(A)$. Note: A is normal means that $C^*(A)$ is unital and commutative.

The plan is to start with $C^*(A)$ commutative unital. Then $\Gamma : C^*(A) \to C(\widehat{C^*(A)})$ is an isometric isomorphism. We need to:

1. Find $\psi : C(\sigma^*(A)) \to C(\widehat{C^*(A)})$ isometric isomorphism (using lemma). It is very important to note that the lemma connects $\widehat{C^*(A)}$ to $\sigma(A)$ *in the algebra* $C^*(A)$. A priori, $\sigma(A)$ in $C^*(A)$(denoted $\sigma^*(A)$) is bigger than $\sigma(A)$ in $\mathcal{A}$.
2. Check that this map ψ does the right thing, on A and 1.
3. Show that $\sigma(A) = \sigma^*(A)$.
4. Define $\Phi_A = \Gamma^{-1} \circ \psi$.

Let's do it:

1. $\gamma \in \widehat{C^*(A)}$, $\gamma \mapsto \gamma(A) \in \sigma^*(A)$ by the lemma. Also, by the lemma, this map is surective. By the definition of weak-$*$ topology on $\widehat{C^*(A)}$, this map is continuous. Want to show it's injective. Well, suppose

   $$\begin{aligned}
   &\gamma_1(A) = \gamma_2(A) \\
   &\gamma_1(A^*) = \overline{\gamma_1(A)} = \overline{\gamma_2(A)} = \gamma_2(A^*) \\
   &\gamma_1(1) = 1 = \gamma_2(1)
   \end{aligned}$$

   which means γ_1, γ_2 agree on $\mathrm{Pol}(A, A^*, 1)$. By continuity, γ_1, γ_2 agree on $C^*(A)$, so $\gamma_1 = \gamma_2$. Also, by the previous day, injective $\Rightarrow$ isometric ($\phi(x-y) = 0 \iff x - y = 0$ by linearly, so taking norms on both sides, $\|\phi(x-y)\| = 0 \iff \|x-y\| = 0$).

   Now, define $\psi : C(\sigma^*(A)) \to C(C^*(A))$. Let $f \in C(\sigma^*(A)), \gamma \in C^*(A)$. Then

   $\psi(f)(\gamma) = f \circ \gamma(A)$, $\qquad \gamma \mapsto f \circ \gamma(A)$ is continuous (need to apply properties of maps)

   Define:
   $$\Phi_A := \Gamma^{-1} \circ \psi.$$

2. let id : $x \mapsto x$ Now,

$$\begin{aligned} \Gamma(A)(\gamma) = \gamma(A) = \mathrm{id}(\gamma(A)) = \psi(\mathrm{id})(\gamma) \\ \Rightarrow \Gamma(A) = \psi(\mathrm{id}) \\ \Rightarrow A = \Psi_A(\mathrm{id}). \\ \Gamma(1)(\gamma) = \gamma(1) = 1 = 1(\gamma(1)) = \psi(1)(\gamma) \\ \Rightarrow \Gamma(1) = \psi(1) \\ \Rightarrow 1 = \Phi_A(1). \end{aligned}$$

3. Want to show $\sigma(A) = \sigma^*(A)$. We know that $\sigma(A) \subset \sigma^*(A)$. (Because $A - \lambda 1$ has access to more inverses in a bigger algebra). Let $\lambda \in \sigma^*(A)$. We will show that $(\lambda 1 - A)$ is not invertible in $\mathcal{A}$.

   Fix $\varepsilon > 0$. Choose $f \in C(\sigma^*(A))$ satisfying $f(x) = 1$, $|f(x)| \leq 1 \quad \forall x$. i.e. $\|f\|_\infty = 1$. $f(x) = 0$ whenever $|x\lambda| > \varepsilon$.

   Now

$$\begin{aligned} \|(A - \lambda 1)\Phi_A(f)\| &= \|\Phi^{-1}(A - \lambda I) f\|_\infty \\ &= \|(\mathrm{id} - \lambda) f\|_\infty \\ &= \sup_{x \in \sigma^*(A)|} |(x - \lambda) f(x)| \\ &= \sup_{|x-\lambda| \leq \varepsilon} |(x - \lambda) f(x)| \leq \varepsilon \|f\|_\infty = 1 \end{aligned}$$

   using that $\|\Phi_A(f)\| = \|f\|_\infty = 1$. so,

$$\|(A - \lambda I)\| < \varepsilon,$$

   meaning if $(A - \lambda I)$ has a set theoretic inverse, $(A - \lambda I)^{-1}$, then $\|(A - \lambda 1)^{-1}\| > 1/\varepsilon$ for all $\varepsilon > 0$. Taking $\varepsilon \to 0$, the norm of the inverse blows up, so it can't be in our C^*-algebra, because its the algebra of bounded operators.

$$\Rightarrow \sigma(A) = \sigma^*(A).$$

Recall:

$$p(A) = \sum_i p(\lambda_i) P_i.$$

$$\Rightarrow \sigma(p(A)) = p(\sigma(A)).$$

Theorem 8 (Spectral mapping)**.** *If $A \in \mathcal{A}$ is normal, $f \in C(\sigma(A))$, then*

$$\sigma(f(A)) = \sigma(\Phi_A(f)) = f(\sigma(A)).$$

Proof. Let $p(x, y) = \sum_{ij} c_{ij} x^i y^j$. Let $\gamma \in \widehat{C^*(A)}$.

$$\gamma(p(A, A^*)) = \sum_{ij} c_{ij} \gamma(A)^i \overline{\gamma(A)}^j = p(\gamma(A), \overline{\gamma(A)}).$$

Now if $f \in C(\sigma(A))$, there is a sequence $p_n(A, A^*) \to f(A)$ (by Stone-Weierstrass, and isometric isomorphism). By continuity of γ, $\gamma(p_n(A, A^*)) \to \gamma(f(A))$. Define

$$p_n^1(x) := p_n(x, \bar{x}).$$

$\Phi_A := \Gamma^{-1} \circ \psi$, so consider $\Phi_A^{-1}(p_n(A, A^*))$.

$$\Gamma(p_n(A, A^*))(\gamma) = \gamma(p_n(A, A^*)) = p_n(\gamma(A, A^*)) = p_n(\gamma(A), \overline{\gamma(A)}) = p_n^1(\gamma(A)).$$

$$\psi(p_n^1)(\gamma) = p_n^1(\gamma(A)) = \Gamma(p_n(A, A^*))(\gamma).$$
$$\Rightarrow \Phi_A^{-1}(p_n(A, A^*)) = p_n^1.$$

taking limits on both sides,
$$f = \Phi_A^{-1}(f(A)) = \lim p_n^1,$$

so $p_n^1 \to f$ uniformly, in particular, it converges pointwise.

$$\forall \gamma \in \widehat{C^*(A)}, p_n^1(\gamma(A)) \to f(\gamma(A)),$$

so
$$f(\gamma(A)) = \gamma(f(A)).$$

$\sigma(f(A)) = \{\gamma(f(A)) : \gamma\} = \{f(\gamma(A)) : \gamma\} = f(\sigma(A)).$ ☐

7.1 Density Matrices and Observables in Infinite Dimensions

Recall, in the finite dimensional case, given an observable and a state, we can define a probability measure on $\sigma(A)$.
$$\text{prob}(\lambda_i) = \text{tr}(\rho P_i).$$

Recall also Other Riez, also known as RRT, also known as Riez-Markov: if X is a locally compact Hausdorff space,

$$\Lambda : C^0(X) \to \mathbb{C}, \text{ linear functional}$$

consider the unique nice Borel measure on X, call it μ, such that

$$\Lambda(f) = \int_X f d\mu.$$

Let $A \in \mathcal{A}$ normal, let's equip ourselves with the functional calculus: $f \in C(\sigma(A))f \mapsto f(A) \in C^*(A)$.

Definition 13. *A* state *is a density matrix equipped with a positive linear functional* $\omega_\rho(A) = tr(\rho A)$, $\omega_\rho \in \mathcal{B}(\mathcal{H})^*$. *In general, we define a* state *to be a positive linear functional on* $\mathcal{A}$ *with norm* 1,
$$\|w\| = \sup_{A \in \mathcal{A}, \|A\|=1} |\omega(A)|.$$

We want to construct a measure that allows us to compute the expected value of an observable. So, we have

$$\Lambda : f \mapsto \omega(f(A)) \in \mathbb{C}.$$

we need to show (next time): ω is continuous $\Rightarrow f \mapsto \omega(f(A)) = \omega \circ \Phi_A(f)$ is continuous. Given a linear functional

$$\Lambda : C(\sigma(A)) \to \mathbb{C}$$

$$\Lambda_{A,\omega}(f) = \omega(f(A)),$$

and applying RRT, there exists $\mu_{A,\omega}$ such that

$$\Lambda_{A,\omega}(f) = \omega(f(A)) = \int_{\sigma(A)} f d\mu_{A,\omega},$$

so

$$\omega(A) = \int_{\sigma(A)} x d\mu_{A,\omega}(x).$$

Definition 14. *$\mu_{A,\omega}$ is called the* spectral measure *of A and ω.*

It satisfies

$$\omega(A) = \int_{\sigma(A)} x d\mu_{A,\omega}(x).$$

We can use the spectral measure to extend the functional calculus to the measurable functions:

$$\int f d\mu_{A,\omega}$$

makes sense if f is borel. Define $\Lambda_{A,\omega}(f) = \omega(f(A))$ for all $\omega \in \mathcal{A}^*$ to reconstruct $f(A)$.

7.2 Positive Elements

Definition 15. *$A \in \mathcal{A}$ is positive if $\sigma(A) \subset [0, \infty)$. Let A^+ be the set of positive elements in $\mathcal{A}$.*

Lemma 4. *Let A be self-adjoint. $A \in \mathcal{A}^+ \iff \|1 - \frac{A}{\|A\|}\| \leq 1$*

Note: this is useful because it's hard to compute the spectrum, easy to compute the norm.

Proof. Suppose $A \in \mathcal{A}^+$.

$$\begin{aligned}
&\Rightarrow \sigma(A) \subset [0, \|A\|] \\
&\Rightarrow \sigma\left(\|1 - \frac{A}{\|A\|}\|\right) \subset [0, 1] \text{ by spectral theorem} \\
&\Rightarrow R\left(\|1 - \frac{A}{\|A\|}\|\right) \leq 1 \Rightarrow \left(\|1 - \frac{A}{\|A\|}\|\right) \leq 1 \text{ spectral radius}
\end{aligned}$$

the other side is not hard. $\square$

Proposition 13 (Useful Proposition). *TFAE*

1. $A \in \mathcal{A}^+$.
2. $A = B^2, B \in \mathcal{A}^+$
3. $A = B^*B$.

Proof.
- (1) $\Rightarrow$ (2), $\sigma(A) \in [0, \infty) \Rightarrow \sqrt{A}$ defined by functional calculus. Take $B = \sqrt{A}$.
- (2) $\Rightarrow$ (3) immediate
- (3) $\Rightarrow$ (1) $B^*B = \Phi_B(\bar{x}x)$.

□

7.3 Putting an order on $\mathcal{A}$

Define $A > B \iff A - B \in \mathcal{A}^+$. We can check that this does define a partial order.

7.4 Approximating the identity

Sometimes we don't want to embed our C^*-algebra into a unital one, instead, we want to approximate the identity.

Proposition 14. *There is a net* $\{E_\alpha\}$ *in* $\mathcal{A}$, $\|E_\alpha\| \leq 1, \alpha \leq \beta$, $E_\alpha \leq A_\beta$. $\lim \|E_\alpha A - A\| = 0, \|1A - A\| = 0$.

Let S be the set of finite subsets of $\mathcal{A}$. Order them with $\subset$. Define

$$f_n(t) = \frac{nt}{1+nt}.$$

$\alpha \in S$, $\alpha = \{A_1, \ldots, A_n\}$. $A_1^2 + \cdots + A_n^2 \in \mathcal{A}^+$. Use functional calculus to get

$$E_\alpha := f_n(A_1^2 + \cdots + A_n^2).$$

8 May 17, 2016

8.1 Review of an Approximate Identity

A *preorder* is a reflexive and transitive relation. We can equip our preorder with:

- *partial order* an order with the anti-symmetric property $a \leq b, b \leq a \Rightarrow a = b$. Note that some elements are not comparable.
- *directed set* $\forall a, b \quad \exists c : a \leq c, b \leq c$.

A total order is both an anti-symmetric set and a directed set.

For example, $\{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$ with $\subseteq$ is not a directed set because $\{1\}, \{3\}$ has no common supset. But it is a partial order.

$\mathbb{R} - \{x_0\}$ with $a \leq b$ if $|x_0 - a| > |x_0 - b|$ (so the further you are from x_0, the bigger you are). This is a directed set, but not a partial order.

8.1.1 Convergence

Definition 16 (Sequence)**.** *A* sequence *is a function* $\mathbb{N} \to X$*, where* X *is a topological space. We say* $\{x_n\}$ *converges to* $x \in X$ *if* $\forall U$ *neighbourhoods to* x*,* $\exists N$ *such that* $n > N \Rightarrow x_n \in U$*.*

Definition 17 (Net)**.** *A* net *is a map from directed set* $D \to X$*, a topological space. We say that* $\{x_\alpha\}_{\alpha \in D}$ *converges to* $x \in X$ *if for all* U *neighbourhood of* x*,* $\exists d \in D : \alpha > d \Rightarrow x_\alpha \in U$*.*

The intuition behind nets is that we wish to generalize notions from metric spaces to topological spaces.

e.g. $f : X \to Y$ if and only if for all nets $\{x_\alpha\}$, $x_\alpha \to x \Rightarrow f(x_\alpha) \to f(x)$. We also get that nice compactness result: X is compact in a topological space if and only if the net $\{x_\alpha\}$ has a convergent subset. Note that this result does not hold for sequences!

8.2 Back to C^*-algebra

We defined A^+ to be the set of positive elements. We put an order relation on $\mathcal{A}$, $A \leq B$ if $B - A \in \mathcal{A}^+$. This is a partial order (*not* a total order).

Why not construct a sequence to approximate the identity? Let $\mathcal{A} \subset \mathcal{A}^1$, the unital C^*-algebra in which $\mathcal{A}$ embeds. We have by default assumed that $\mathcal{A}$ is not unital. Suppose $E_n \to 1 \in \mathcal{A}^1$, such that $E_n \in \mathcal{A}$. Then $\{E_n\}$ is Cauchy $\Rightarrow 1 \in \mathcal{A}$ by completeness, a contradiction.

8.3 Approximation of the Identity (Construction)

Let ζ be all finite subsets of $\mathcal{A}$, ordered by $\subseteq$. This is a directed set. We use functional calculus $f_n(t) = \frac{nt}{1+nt}$, $\alpha := \{A_1, \ldots, A_n\} \in S$.

$$E_\alpha := f_n(A_1^2 + \ldots A_n^2)$$

Let $\alpha \in S : \alpha \mapsto E_\alpha \in \mathcal{A}$. We found $\|E_\alpha\| \leq 1$. $\alpha \leq \beta \Rightarrow E_\alpha \leq E_\beta$ on $\mathcal{A}$. Then, $\|E_\alpha A - A\| \to 0$ and $\|AE_\alpha - A\| \to 0$.

8.4 States

In finite dimensions, $\mathcal{A} = \mathcal{B}(\mathcal{H})$. our states are given by density matrices $\rho \in \mathcal{B}(\mathcal{H})$, $\rho \leq 0$, $\text{tr}(\rho) = 1$. $\text{tr}(\rho A)$ is interpreted as the expected value of A in ρ.

It is impractical to generalize this notion naively in infinite dimensions, as not many operators are trace class. Instead, we interpret states as operators acting on the observables:

We may define a linear functional $\omega_\rho(A) = \text{tr}(\rho A)$, $\omega_\rho : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$. The map is positive: it maps positive operators to $\mathbb{R}^+ \subset \mathbb{C}$.

We can define a norm on $\mathcal{B}(\mathcal{H})^*$.

$$\|\omega\| = \sup_{A \in \mathcal{B}(\mathcal{H}), \|A\|=1} |\omega(A)|.$$

Then, $\mathrm{tr}(\rho) = 1 \Rightarrow \|\omega_\rho\| = 1$. Indeed,

$$\begin{aligned}\mathrm{tr}(\rho A) &= \sum_i \langle \psi_i \rho A \psi_i \rangle \\ &= \sum_i \lambda_i \langle \psi_i \rho \psi_i \rangle \\ &\leq \sum_i |\lambda_i| \langle \psi_i \rho \psi_i \rangle \leq \|A\| \mathrm{tr}(\rho) \Rightarrow \|\omega\| \leq 1.\end{aligned}$$

So to summarize, we have a correspondance between $\rho \to \omega_\rho$ such that if $\mathrm{tr}(\rho) = 1$ and $\rho \geq 0$, the corresponding ω_ρ is positive, as has $\|w\| = 1$.

Definition 18. *A linear functional over $\mathcal{A}$ is* positive *if $\omega(AA^*) \geq 0$, $\forall A \in \mathcal{A}$. A positive linear functional with norm 1 is called* a state*.*

Physically, $\omega(A)$ is the expected value of A in ω.
We will try to show the following:
A linear functional is continuous $\iff$ bounded $\iff$ positive.

Proposition 15. *Let X, Y be norm spaces. Let $L : X \to Y$ be linear. L is bounded $\iff$ it is continuous.*

Proof. Suppose L is bounded, i.e. $\|Lx\|_Y \leq M\|x\|_X$. Then,

$$\|L(x+h) - L(x)\| = \|L(h)\|_Y \leq M\|h\|_X.$$

$\|h\| \to 0 \Rightarrow \|L(x+h) - L(x)\| \to 0 \Rightarrow L$ is continuous at x. This actually gives Lipchitz continuity!

Conversely, suppose L is continuous. Actually, we only need continuity at 0! Let $\varepsilon = 1$. Then we can find δ such that $\|L(h)\| = \|L(h) - L(0)\| < 1$ whenever $\|h - 0\| = \|h\| < \delta$. Fix an arbitrary $x \in X$. Then

$$\left\| \delta \frac{x}{\|x\|} \right\| = \delta.$$

$$\|Lx\|_Y = \left\| \frac{\|x\|}{\delta} L(\frac{\delta}{\|x\|} x) \right\| \leq \frac{\|x\|}{\delta} = \underbrace{\frac{1}{\delta}}_{M} \|x\|.$$

□

Proposition 16. *If ω is linear on $\mathcal{A}$ (a Banach space with an algebra, and the inequality $\|AB\| \leq \|A\|\|B\|$) and ω is positive on $\mathcal{A}$, then ω is bounded.*

Proof. Let $\{A_i\}$ be a sequence in $\mathcal{A}^+$, $\{\lambda_i\}$ a sequence of positive numbers in $\mathbb{R}$ such that $\sum_i \lambda_i < \infty$, $\|A_i\| \leq 1$. Then,

$$\left\| \sum_{i=1}^m \lambda_i A_i - \sum_{i=1}^n \lambda_i A_i \right\| = \| \sum_{i=n}^m \lambda_i A_i \| \leq \sum_{i=n}^m \lambda_i$$

so, $\sum_{i=1}^{\infty} \lambda_i A_i$ converges in $\mathcal{A}$. In fact, it converges in $\mathcal{A}^+$ (check this).

Call the limit A. Then, $\sum_i \lambda_i \omega(A_i) = \omega(\sum_i \lambda_i A_i) \leq \omega(A) < \infty$. Define:

$$M := \sup\{\omega(A) : A \geq 0, \|A\| \leq 1\} < \infty.$$

(We will not prove this.) To complete our proof, note that

$$A = \frac{A + A^*}{2} + i\frac{A - A^*}{2i},$$

the sum of two self-adjoint operators. Now,

$$A_{re} = \frac{|A_{re}| + A_{re}}{2} - \frac{|A_{re}| - A_{re}}{2}.$$

$\Rightarrow \|\omega\| = \sup\{\omega(A) : A \in \mathcal{A}, \|A\| \leq 1\} \leq 4M < \infty.$ $\square$

8.5 Geometry of States

We will do geometry on $\mathcal{B}(\mathcal{H})^*$ instead of $\mathcal{H}$, by using the induced geometry of the spaces $\mathcal{H}, \mathcal{B}(\mathcal{H})$. This corresponds to doing geometry on $\mathcal{A}^*$, the topological dual of $\mathcal{A}$.

Proposition 17 (Cauchy-Schwarz)**.** *Let $\omega \in \mathcal{A}^*$. Then*

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$$

Proof. Consider $\lambda \in \mathbb{C}$, $\omega((\lambda A + B)^*(\lambda A + B)) \geq 0$. By linearly,

$$|\lambda|^2\omega(A^*A) + \bar{\lambda}\omega(A^*B) + \lambda\omega(B^*A) + \omega(B^*B) \geq 0.$$

Since we know we have a real quantity,

$$\omega(A^*B) = \overline{\omega(B^*A)}.$$

Let $\lambda = a + ib, a, b \in \mathbb{R}$. Then

$$f(a, b) := a^2\omega(A^*A) + 2a\text{Re }(\omega(B^*A)) - 2b\text{Im }(\omega(B^*A)) + b^2\omega(A^*A) + \omega(B^*B) \geq 0$$

Since $f : \mathbb{R}^2 \to \mathbb{R}$. With some calculus, and calculating some partials, we get $f(a, b)$ is positive definite everywhere, and it has one local min:

$$a_{\min} = \frac{-\text{Re }(\omega(B^*A))}{\omega(A^*A)}, \qquad b_{\min} = \frac{\text{Im }(\omega(B^*A))}{\omega(A^*A)}$$

then $f(a, b) \geq 0 \iff f(a_{\min}, b_{\min}) \geq 0$. $\square$

Lemma 5. $\|\omega\| = \sup_\alpha \omega(E_\alpha^2)$.

Reminder: E_α is the approximate identity, with $\|E_\alpha\| \leq 1, E_\alpha \geq 0$.

Proof. Cauchy-Schwarz ($\Rightarrow$)

$$|\omega(AE_\alpha)|^2 \leq \omega(A^*A)\omega(E_\alpha^2) \leq M\|A\|\omega(E_\alpha^2)$$

take the limit in α, and use the continuity of ω.

$$|\omega(A)|^2 \leq M\|A\|^2 \sup_\alpha \omega(E_\alpha^2)$$

$$\Rightarrow \|\omega\|^2 \leq M \sup_\alpha \omega(E_\alpha^2) \leq \|\omega\| \sup_\alpha \omega(E_\alpha^2)$$

where $M := \sup\{\omega(A) : A \geq 0, \|A\| \leq 1\} \leq \|\omega\|$

$$\Rightarrow \|\omega\| \leq \sup_\alpha \omega(E_\alpha^2).$$

For each $\alpha, \|E_\alpha^2\| \leq 1 \Rightarrow \sup_\alpha \omega(E_\alpha^2) \leq \|\omega\|$. □

9 May 18, 2016

Last time: we talked about generalized states in the operator algebra setting. We said a state was a norm 1 positive functional in our operator algebra. By positivity, we mean positivity of the spectrum, or that $\omega(A^*A) \in \mathbb{R}^+ \quad \forall A$.

- For linear operators, continuous $\iff$ bounded.
- For linear functionals, positive $\Rightarrow$ bounded $\iff$ continuous. That means that the set of states is S^1 in $\mathcal{A}^*$.

Regarding the geometry of the states,

- We proved a version of Cauchy-Schwarz, namely $|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)$.
- We used the approximate identity to show that $\|\omega\| = \sup_\alpha \omega(E_\alpha^2)$.

As a corollary, states satisfy the triangle *equality*!

Corollary 1. $\|\omega_1+\omega_2\| = \sup_\alpha \omega_1(E_\alpha^2)+\omega_2(E_\alpha^2) = \sup_\alpha \omega_1(E_\alpha^2)+\sup_\alpha \omega_2(E_\alpha^2) = \|\omega_1\|+\|\omega_2\|$.

Corollary 2. *If* $\|\omega_1\| + \|\omega_2\| = 1$, *then* $\|\lambda\omega_1 + (1-\lambda)\omega_2\| = \lambda\|\omega_1\| + (1-\lambda)\|\omega_2\| = 1$.

This means the set of states is convex.

Example 1 (Bloch Sphere)**.**

Let ρ be a density matrix on $\mathcal{H} = \mathbb{C}^2$. This space has basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where the last three matrices are trace 0. Then $\rho = \mu 1 + \vec{\lambda} \cdot \vec{\sigma}$, and since $\mathrm{tr}(\rho) = 2\mu = 1$ we have $\mu = 1/2$. We may now compute the eigenvalue of ρ in terms of $\vec{\lambda}$. We get $\frac{1}{2}(1 \pm \|\vec{\lambda}\|)$.

Now, $\rho \geq 0 \Rightarrow \|\vec{\lambda}\| \leq 1$. This allows us to define a map $\rho \mapsto \vec{\lambda}$. This preserves geometry in the sense that the states ρ will be mapped to 3-point $\vec{\lambda}$, which is inside the unit sphere.

Recall that a state is pure if $\mathrm{tr}(\rho^2) = 1$. This means that $p_i = 1$ for exactly one i in the decomposition $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$. Since $\mathrm{tr}(\rho^2) = \frac{1}{2}(1 + \|\vec{\lambda}\|^2)$, we get that $\|\vec{\lambda}\| = 1$, and so the pure states are the extremal points of the set of states.

Also, recall that $\|\omega\| = \sup_\alpha \omega(E_\alpha^2)$.

Proposition 18. $\|\omega\| = \omega(1)$ *(If $\mathcal{A}$ is unital).*

Proof. $\|1 - E_\alpha^2\| \leq \|1 - E_\alpha\| + \|E_\alpha - E_\alpha^2\| \leq \underbrace{\|1 - E_\alpha\|}_{\to 0} + \|E_\alpha\| \underbrace{\|1 - E_\alpha\|}_{\to 0}$. so $\omega(1 - E_\alpha^2) \leq \|\omega\| \|1 - E_\alpha\|^2 \to 0$. □

9.1 Measurement

Recall that for a linear functional $C(\sigma(A)) \to f(A)$, we can define for each A, ω

$$\Lambda_{A,\omega} : f \mapsto \omega(f(A)).$$

By the Riesz representation,

$$\Lambda_{A,\omega}(f) = \int_{\sigma(A)} f d\mu_{A,\omega}.$$

where $\mu_{A,\omega}$ is the spectral measure.

We verify this matches our finite setting:

$$A = \sum_i \lambda_i P_i,$$

$$\mathrm{prob}(\lambda_i) = \mathrm{tr}(\rho P_i).$$

Define the characteristic function: $1_{\{\lambda_i\}}(x) = \begin{cases} 1 & x = \lambda_i \\ 0 & \neq \lambda_i \end{cases}$. Then $1_{\{\lambda_i\}}(A) = P_i$.

If we take $E \subset \sigma(A)$, then

$$1_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}.$$

We may define $1_E(A)$ via the functional calculus. Now by Riesz,

$$\omega(1_E(A)) = \int_{\sigma(A)} 1_E(x) d\mu_{A,\omega}(x) = \int_E d\mu_{A,\omega}$$

Physically, we interpret $\int_E d\mu_{A,\omega}$ to be the probability of the result of the measurement to be in E. We will use this as our axiom for the probability of the result to be in E.

Note that in particular, $\omega(A) = \omega(\mathrm{id}(A)) = \int_{\sigma(A)} x d\mu_{A,\omega}(x)$. This is no different than a classical probability set-up. We stress that it takes *multiple* measurements for the non-commutative properties of quantum mechanics to manifest themselves.

9.2 Unbounded Operator

Let X be a Banach space.

Definition 19. *Let $D(T) \subset X$ be the* domain *of T in X, where the inclusion is a subspace inclusion. An* operator *is a linear map $T : D(T) \to X$.*

Before, we had

$$\|T\|_{\text{op}} = \sup_{\|x\|=1, x \in X} \|Tx\|$$

now we'll define

$$\|T\|_{\text{op}} = \sup_{\|x\|=1, x \in D(T)} \|Tx\|$$

Note that this operator norm may not be finite.

An example of such a setup would be the space $X = C^0([0,1])$, with L^2-norm. Let

$$T : D(T) \to X,$$

$$f(x) \mapsto \frac{d}{dx} f(x)$$

Let $f_n(x) = \sin(2\pi nx)$. Then $\|f_n\|_2 = \frac{1}{\sqrt{2}}$, $\|Tf_n\|_2 = \frac{2\pi n}{\sqrt{2}}$, so

$$\|T\|_{\text{op}} \geq \sup_n \frac{\|Tf_n\|_2}{\|f_n\|_2} = \sup_n 2\pi n = \infty.$$

Definition 20. *If $\|T\|_{op} < \infty$, then T is bounded. Otherwise, it is* unbounded.

If $D(T)$ is dense in X, and T is bounded, we can always constrict a bounded extension to T. Note: bounded is equivalent to continuous. This is because we can define the extension

$$\tilde{T}(x) = \lim_n T(x_n).$$

Then $\tilde{T}$ is continuous by construction, and hence bounded.

Definition 21. *Let T be an operator. The* graph *of T is the set of pairs*

$$\Gamma(T) = \{(x, Tx) : x \in D(T)\}. \subset X \times X$$

If $\Gamma(T)$ is closed, we say that T is *closed.* Another way to put it is to say if $(x_n, Tx_n) \to (x, y)$ for $x \in D(T)$, then $Tx = y$. This is some sense of 'almost continuity', because we only require for T to be continuous on its domain. For example, $f(x) = 1/x$ is closed on the domain $\mathbb{R} - \{0\}$.

Theorem 9 (Closed Graph)**.** *If $D(T) = X$, and T is closed, then T is bounded.*

10 May 25, 2016

10.1 Dynamics

We are developing this theory of unbounded operators, because we want to be able to deal with a large reservoir of infinite energy. By the relation between the norm and the spectral radius, we know that this means that the norm of our Hamiltonian will need to blow up. So, we need to be able to deal with unbounded operators.

10.1.1 Finite-dimensional setting

$$\tau^t(A) = e^{itH} A e^{-itH}$$

for H a self-adjoint bounded operator.

We want to rewrite this, because we don't like sandwiching operators. Taking derivatives carefully (i.e. being aware that things don't commute):

$$\begin{aligned}
&\frac{d}{dt}\tau^t(A) = iHe^{itH}Ae^{-itH} + e^{itH}Ae^{-itH}(-iH) \\
&= iH\tau^t(A) - i\tau^t(A)H \\
&= i[H, \tau^t(A)]
\end{aligned}$$

$\tau^0(A) = A$. Solving this IVP,

$$\tau^t(A) = e^{it[H,\cdot]}A$$

where

$$\begin{aligned}
[H,\cdot] : \mathcal{B}(\mathcal{H}) &\to \mathcal{B}(\mathcal{H}) \\
A &\mapsto [H, A]
\end{aligned}$$

Note that

$$e^{it[H,\cdot]}A \neq e^{it[H,A]},$$

which we can check with the power series expansion. We can check our solution:

$$\begin{aligned}
\frac{d}{dt}(e^{it[H,\cdot]}A) &= \frac{d}{dt}\left(\sum_{n=0}^{\infty} \frac{(it)^n([H,\cdot])^n A}{n!}\right) \\
&= \frac{d}{dt}\left(\sum_{n=0}^{\infty} \frac{(it)^n([H,[H,\ldots[H,A]]])}{n!}\right) \\
&= \left[iH, \sum_{n=1}^{\infty} \frac{(it)^{n-1}}{(n-1)!}[H,\cdots[H,[H,a]]\right] \\
&= i[H, e^{it[H,\cdot]}A]
\end{aligned}$$

We are going to define

$$\partial_H := i[H, \cdot]$$

and call it the *generator of our dynamics*.

Remark 1. *If H is self-adjoint, then e^{itH} is unitary.*

$$(e^{itH}\phi, e^{itH}\psi) = (\phi, e^{-itH}e^{itH}\psi) = (\phi, \psi)$$

The operation $A \mapsto UAU^*$, where U is unitary preserves the spectrum.

$$\begin{aligned} U(\lambda I - A)U^* &= \lambda I - UAU^* \\ &= U^*(\lambda I - UAU^*)U = \lambda I - A \end{aligned}$$

$\Rightarrow (\lambda I - A)$ is invertible $\iff \lambda I - UAU^*$ is invertible.

Also, notice that if $A = \sum_i \lambda_i P_i$, then $UAU^* = \sum_i \lambda_i UP_iU^*$ notice that $(UP_iU^*)(UP_iU^*) = UP_iU^*$ so we already have the spectral decomposition. Then,

$$\text{prob}_\rho(\lambda_i) = \text{tr}(\rho P_i) \to \text{prob}_\rho^t(\lambda_i) = \text{tr}(\rho UP_iU^*)$$

so the weight associated with each λ_i changes, but

$$\sum_i (\text{tr}\rho UP_iU^*) = \text{tr}(\rho U(\sum_i P_I)U^*) = \text{tr}(\rho) = 1$$

Theorem 10 (Stone Theorem)**.** *If $t \mapsto U(t)$ is 'nice', then $U(t) = e^{itA}$ for A self-adjoint.*

The conditions we need for Stone are

1. Strong continuity: $\forall \psi \in \mathcal{H}$, $t \mapsto U(t)\psi$ is continuous with respect to $\|\cdot\|_{\mathcal{H}}$. (Note this is actually not super strong).

2. $U(t+s) = U(t)U(s)$, $(\mathbb{R}, +)$ is a group, and $(U(t), \cdot)$ inherits this group structure. We do have that here, because $e^{i(t+s)A} = e^{itA}e^{isA}$.

We will generalize this even more!

- Notice that for any unitary U,

  $$\begin{aligned} &\mathcal{C} : \mathcal{A} \to \mathcal{A}, \\ &A \mapsto UAU^* \end{aligned}$$

  has a linear structure (actually, it preserves all the algebraic structure because we have a surjective isometry),

  $$\|UAU^*\| \leq \|U\|\|U^*\|\|A\| \leq \|A\|.$$

  $\mathcal{C}$ is an isomorphism from $\mathcal{A} \to \mathcal{A}$, which is called an *automorphism.*

Definition 22. *A $\mathcal{C}^*$-dynamics is a map $t \in \mathbb{R}, t \mapsto \tau^t \in Aut\ (\mathcal{A})$, which satisfies $\tau^{t+s} = \tau^t\tau^s$ and $t \mapsto \tau^t$ strongly continuous (in another way than in the previous!), with $\tau^0 = id$.*

The reason why this strongly continuous is not the same as the previous is because we now have three layers of objects. We have the Hilbert space with a norm, and attached to that we have this space of bounded operators $\mathcal{B}(\mathcal{H})$ which acts on the Hilbert space, and a space of linear functionals $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ which acts on those operators. There is strongly continuous in the sense of $\mathcal{B}(\mathcal{H})$, and strongly continuous in the sense of $\mathcal{L}(\mathcal{B}(\mathcal{H}))$, which is the one we just mentioned. Strongly continuous on the bottom level does not induce strong continuity on top.

Theorem 11 (Hille-Yosida)**.** *$t \mapsto \tau^t$ be a $\mathcal{C}^*$-dynamic. Then, let*

$$D(\delta) := \{A \in \mathcal{A} : \lim_{t\to 0} \frac{\tau^t(A) - A}{t} \text{ exists. } \}$$

$$\delta(A) = \lim_{t\to 0} \frac{1}{t}(\tau^t(A) - A) \qquad A \in D(\delta)$$

1. *$D(\delta)$ is dense in $\mathcal{A}$.*
2. *δ is closed.*
3. *If $A \in D(\delta)$, then $\tau^t(A) \in D(\delta)$ and we have the following,*

$$\frac{d}{dt}\tau^t(A) = \delta(\tau^t(A)) = \tau^t(\delta(A))$$

$$\frac{d}{dt}\tau^t(A) = i[H, \tau^t(A)] = \delta_H(\tau^t(A)).$$

Lemma 6.

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \tau^s(A)ds = \tau^t(A)$$

Proof(ish).

$$\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}\int_0^{t+\varepsilon} \tau^s(A)ds - \int_0^t \tau^s(A)ds\right]$$
$$= \frac{d}{dt}\int_0^t \tau^s(A)ds = \tau^t(A)$$

where to use something similar to the fundamental theorem of calculus, we have used the property that $t \to \tau^s$ is strongly continuous. ∎

Now we are ready to prove Hille-Yosida.

Proof of Hille-Yosida. 1. Define

$$R_\varepsilon := \frac{1}{\varepsilon}\int_0^\varepsilon \tau^s ds.$$

For any $A \in \mathcal{A}$, we claim that

$$\frac{1}{t}\int_0^t \tau^s(A)ds \in D(\delta)$$

Indeed,

$$\frac{1}{h}(\tau^h - \mathrm{id})\int_0^t \tau^s(A)ds = \frac{1}{h}\int_t^{t+h} \tau^s(A)ds - \frac{1}{h}\int_0^h \tau^s(A)ds$$

Now take $h \to 0$, so then

$$\delta(\int_0^t \tau^s(A)ds) = \tau^t(A) - \tau^0(A) = \tau^t(A) - A.$$

In particular, $\int_0^t \tau^s(A)ds \in D(\delta)$. (This is a good general technique for figuring out if something is in $D(\delta)$. Oftentimes, it is best to just compute it.) So, for all $A \in \mathcal{A}$, $R_\varepsilon(A) \in D(\delta)$.

Now we want to compute the strong $\lim_{\varepsilon\to 0} R_\varepsilon$.

$$\lim_{\varepsilon\to 0} R_\varepsilon(A) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon \tau^s(A)ds = \tau^0(A) = A,$$

so $s - \lim_{\varepsilon\to 0} R_\varepsilon = \mathrm{id}$.

Since Dom (id) $= \mathcal{A}$,

$$R_\varepsilon(\mathcal{A}) \subseteq D(\delta), \qquad \forall \varepsilon > 0.$$

fix $A \in \mathcal{A}, \eta > 0$, then for small enough ε, $\| \underbrace{R_\varepsilon(A)}_{D(\delta)} - A\| < \eta \Rightarrow D(\delta)$ is dense.

2. δ is closed. We want to show that if $A_n \to A$, and $\delta(A_n) \to B$, then $\delta(A) = B$.

$$\begin{aligned}\frac{1}{t}(\tau^t - \mathrm{id})A_n &= \frac{1}{t}\delta(\int_0^t \tau^s(A_n)ds)\\ &= \frac{1}{t}\int_0^t \tau^s(\delta(A_n))ds\end{aligned}$$

evaluate LHS, RHS as $n \to \infty$, and get

$$\frac{1}{t}(\tau^t - \mathrm{id})A = \frac{1}{t}\int_0^t \tau^s(B)ds$$

now taking $t \to 0$, we get

$$\delta(A) = B$$

3. $A \in D(\delta)$,

$$\frac{1}{h}(\tau^h - \mathrm{id})\tau^t(A) = \frac{1}{h}\tau^t((\tau^h - \mathrm{id})(A))$$

taking $h \to 0$ on both sides,

$$\delta(\tau^t(A)) = \tau^t(\delta(A))$$

$$
\begin{aligned}
\frac{d}{dt}\tau^t(A) &= \lim_{h\to 0}\frac{1}{h}(\tau^{t+h}(A)-\tau^t(A)) \\
&= \lim_{h\to 0}\frac{1}{h}(\tau^t(\tau^h(A)-A)) \\
&= \tau^t(\delta(A)) \\
&= \delta(\tau^t(A))
\end{aligned}
$$

so in some sense,

$$''\tau^t = e^{t\delta}"$$

□

10.2 Equilibrium States

In the finite setting,

$$\rho_\beta = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})},$$

Proposition 19. *A state $\omega(\cdot) = tr(\rho\cdot)$ is a Gibbs state at temperature β^{-1} if and only if*

$$\omega(A\tau^{i\beta}(B)) = \omega(BA)$$

for all $A, B \in \mathcal{B}(\mathcal{H})$. This condition is called the KMS condition.

Proof. Suppose ω is Gibbs. Then

$$\omega(\tau^{i\beta}(B)) = \mathrm{tr}(\frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}Ae^{-\beta H}Be^{\beta H}) = \mathrm{tr}(\frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}BA) = \omega(BA)$$

Conversely, suppose ω is β-KMS, then for all A, B,

$$\mathrm{tr}(\rho BA) = \mathrm{tr}(\rho Ae^{-\beta H}\beta e^{\beta H}).$$

Let $X = e^{\beta H}\rho$, $Y = Ae^{-\beta H}$, then

$$\mathrm{tr}(XBY) = \mathrm{tr}(XYB) = \mathrm{tr}(BXY), \qquad \forall Y, B$$

$$
\begin{gathered}
\mathrm{tr}([X,B]Y) = 0 \\
\Rightarrow [X,B] = 0 \qquad \forall B \\
X = \alpha I \Rightarrow e^{\beta H}\rho = \alpha I.
\end{gathered}
$$

□

11 May 26, 2016

Last time, we saw that a $\mathcal{C}^*$ dynamics is a map $\mathbb{R} \to \text{Aut }(\mathcal{A}) \subset \mathcal{B}(\mathcal{A}), t \to \tau^t$, satisfying

1. $\tau^{t+s} = \tau^t \tau^s$
2. $t \mapsto \tau^t$ is strongly continuous, $\forall A \in \mathcal{A}, \quad t \mapsto \tau^t(A)$ is continuous.
3. τ^t is an automorphism for all t.

and the Hille-Yosida theorem, which state that the domain of the operator

$$\delta := \lim_{h \to 0} \frac{1}{h}(\tau^t(A) - A)$$

is dense in $\mathcal{A}$. δ is also a close operator, and satisfies the differential equation

$$\frac{d}{dt}\tau^t(A) = \delta\tau^t(A).$$

We use the notation $\tau^t = e^{t\delta}$ and only say δ generates τ^t. Moreover, $\tau^t(A) \in D(\delta)$ if $A \in D(\delta)$, and $\tau^t(A)$ is differentiable if $A \in D(\delta)$.

In the finite setting, $\tau^t(A) = e^{itH}Ae^{-itH}, \quad \delta_H(A) = i[H, A]$, $\tau^{i\beta}(A) = e^{-\beta H}Ae^{\beta H}$ and the equilibrium states are

$$\rho = \frac{e^{-\beta H}}{\text{tr}(e^{-\beta H)}}$$

so we see that there is only one equilibrium state per temperature, which does not capture reality when we have a very big reservoir.

Proposition 20. *A state ω is β-Gibbs $\iff \omega(A\tau^{i\beta}(B)) = \omega(BA), \quad \forall A, B \in \mathcal{A}$.*

Remark 2. *We can rewrite KMS as $\omega(A\tau^{t+i\beta}) = \omega(\tau^t(B)A) \quad \forall t$.*

we may define

$$F_{A,B}(z) := \omega(A\tau^z(B)).$$

At the infinite temperature limit, $\beta \to 0$, we want

$$\omega(\tau^t(B)A) = \omega(A\tau^t(B))$$

this is a trace state. If the dimension of $\mathcal{H}$ is n, then $\omega(A) = \frac{1}{n}\text{tr}(A)$, and the state is given by

$$\rho = \frac{1}{n}I.$$

This ρ is the most chaotic state, in the sense it has the most entropy, which makes sense because at infinite temperature, you expect to have a very chaotic situation. To solve for this ρ, we have set:

$$S(\rho) = -\sum_{i=1}^{n} \lambda_i \log \lambda_i,$$

where λ_i are the eigenvalues of ρ. Restricting to states gives the constraint $\sum_{i=1}^n \lambda_i = 1$. By Lagrange multipliers, the extremized state is the ρ we obtained.

11.1 Generalizing the KMS condition

Problem: for an abstract dynamic, τ^t, $\quad t \mapsto \tau^t(A)$ needs to be extended to $\mathbb{C}$. We'd like in particular for the extension to be analytic. Unfortunately, we can't do this for all the elements of the $\mathcal{C}^*$-algebra, but there is a sufficiently nice subset of our $\mathcal{C}^*$-algebra $\mathcal{A}$ for which this applies. This subset will be dense in $\mathcal{A}$.

Definition 23. *Fix τ^t on $\mathcal{A}$. Then $A \in \mathcal{A}$ is* analytic *for τ^t if there exists*

$$I_\lambda = \{z \in \mathbb{C} : |Im\ z| < \lambda\}$$

and a function $f : I_\lambda \to \mathcal{A}$ such that

1. *$f(t) = \tau^t(A), \quad t \in \mathbb{R}$*
2. *$\omega(f(z))$ is an analytic function $\forall \omega \in \mathcal{A}^*$.*

Proposition 21. *If A is analytic for τ^t, then $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists.*

To prove this, we need uniform boundedness.

Theorem 12. *Let X be a Banach Space, Y normed space, and $\mathcal{Y}$ is a family of bounded linear operators $X \to Y$.*

Then, if

$$\sup\{\|Tx\| : T \in \mathcal{Y}\} < \infty, \qquad \forall x \in X,$$

we have

$$\sup\{\|T\|_{\text{op}} : T \in \mathcal{Y}\} < \infty$$

Proof of f being analytic. Let C denote a circle inside I_λ, centered at z, and K to be a disk inside the circle, and g, h, two different small complex quantities. We use the Cauchy integral formula to write

$$\omega(f(z)) = \frac{1}{2\pi i}\int_C \frac{\omega(f(y))}{y-z}dy$$

assume $z+h$ and $z+g \in K$. Then

$$\begin{aligned}
&\frac{1}{h-g}\left(\frac{1}{h}\left[\omega(f(z+h)) - \omega(f(z))\right] - \frac{1}{g}\left[\omega(f(z+g)) - \omega(f(z))\right]\right) \\
&= \frac{1}{(h-g)2\pi i}\int_C \omega(f(y))\left[\frac{1}{h(y-z-h)} - \frac{1}{g(y-z-g)} - \frac{1}{h(y-z)} + \frac{1}{g(y-z)}\right] \\
&= \frac{1}{2\pi i}\int_C \omega(f(y))\underbrace{(y-z-g)^{-1}(y-z-h)^{-1}}_{y-(z+g)\text{ is bounded}}(y-z)^{-1}dy
\end{aligned}$$

This whole thing is bounded uniformly in g and h. We will use uniform boundedness. let

$$\Lambda_{h,g} : \mathcal{A}^* \to \mathbb{C}$$

$$\omega \mapsto \frac{1}{h-g}\left(\frac{1}{h}[\omega(f(z+h)) - w(f(z))]\dots\right)$$

Now,

$$\begin{aligned}
&\sup\{|\Lambda_{h,g}(\omega)| : h, g \text{ small }\} < \infty \quad \forall \omega \\
\Rightarrow &\{\|\Lambda_{h,g}\| : h, g \text{ small }\} < \infty \\
&\|\Lambda_{h,g}\| = \sup_{|\omega|=1} \frac{1}{|h-g|}\left\|\frac{1}{h}\left[\omega(f(z+h)) - \omega(f(z))\right] - \frac{1}{g}\left[\omega(f(z+g)) - \omega(f(z))\right]\right\| \\
&= \sup_{\|\omega\|=1} \frac{1}{|h-g|}\left\|\frac{1}{h}\left[f(z+h)) - f(z))\right] - \frac{1}{g}\left[f(z+g) - f(z)\right]\right\| \|\omega\| \\
&= \sup_{\|\omega\|=1} \frac{1}{|h-g|}\left\|\frac{1}{h}\left[f(z+h)) - f(z))\right] - \frac{1}{g}\left[f(z+g) - f(z)\right]\right\| < \gamma
\end{aligned}$$

which is uniform on g and h. This implies

$$\left\|\frac{1}{h}(f(z+h) - f(z)) - \frac{1}{g}(f(z+g) - f(z))\right\| < |h-g|\gamma$$

as we pick g, h very small, the LHS quantity becomes very small. We know the limit exists because we can make the thing into a Cauchy sequence, and the space is complete. □

11.2 Defining Integrals

Proposition 22. *Let $A \in \mathcal{A}$, μ measure on $\mathbb{R}$. There exists a B such that*

$$\omega(B) = \int \omega(\tau^t(A))d\mu(t) \qquad \forall \omega \in \mathcal{A}^*$$

note the ω's. Like the previous, this is because we want to translate analytic notions in $\mathbb{C}$ to notions in the $\mathcal{C}^*$-algebra, and the way we do it is by evaluating our operators with these functions ω to state the notions.

Fittingly, we denote B by $\int \tau^t(A)d\mu(t)$ when the above condition is satisfied.

11.3 Sketchy Sketch of Density of Analytic Elements

Fix $A \in \mathcal{A}$. Define

$$A_n = \sqrt{\frac{n}{\pi}} \int \tau^t(A)e^{-nt^2}dt$$

Claim 1. *A_n is analytic.*

$$
\begin{aligned}
f_n(z) &= \sqrt{\frac{n}{\pi}} \int \tau^t(A) e^{-n(t-z)^2} dt \\
f_n(s) &= \sqrt{\frac{n}{\pi}} \int \tau^t(A) e^{-n(t-s)^2} dt \\
&= \sqrt{\frac{n}{\pi}} \int \tau^{t+s}(A) e^{-nt^2} dt \\
&= \sqrt{\frac{n}{\pi}} \tau^s \underbrace{\int \tau^t(A) e^{-nt^2} dt}_{A_n}
\end{aligned}
$$

Definition 24. *$\mathcal{A}$ is a $\mathcal{C}^*$-algebra, ω is a KMS state if*

$\omega(A\tau^{i\beta}(B)) = \omega(BA), \qquad \forall A, B$ *in a norm-dense subset of the algebra of the τ^t-analytic elements.*

We will call the set in which A, B belongs to $\mathcal{A}_{KMS}$. Gibbs $\Rightarrow$ KMS still holds under this condition and $\not\Leftarrow$ Gibbs. This is good because we wanted more equilibrium states.

11.4 Properties of KMS states

Proposition 23. *If ω is $\beta - \tau$ KMS, then $\omega(\tau^t(A)) = \omega(A) \qquad \forall A \in \mathcal{A}, t \in \mathbb{R}$.*

Proof. W.l.o.g, let $\beta = -1$ (this won't be explained. It's not obvious). Let $B \in \mathcal{A}_{KMS}$.

$$F(z) = \omega(\tau^z(B))$$

is analytic, since β is analytic. This is bounded on the strip

$$I = \{z \in \mathbb{C} : -1 \leq \text{Im }(z) \leq 0\},$$

as

$$|F(z)| \leq \|\tau^z(B)\| = \|\underbrace{\tau^{\text{Re }(z)}}_{\text{Aut}} \tau^{i\text{Im }(z)}(B)\| = \|\tau^{i\text{Im }(z)}(B)\|$$

(we will prove the group law for complex numbers later). Now,

$$\sup_{z \in I} |F(z)| \leq \sup_{\lambda \in [-1,0]} \|\tau^{i\lambda}(B)\|$$

Note that $\lambda \mapsto \|\tau^{i\lambda}(B)\|$ is continuous. We are on a compact set, so the sup is achieved, and $\sup_{z\in I} |F(z)| < \infty$. $F(z)$ is periodic on i,

$$F(z - i) = \omega(I\tau^i(\tau^z(B))) = \omega(\tau^z(B)) = F(z)$$

hence, F is bounded on $\mathbb{C}$, and analytic. By Liouville, F is constant. $\square$

Note this the property that we want! Since KMS is a generalization of the Gibbs state, we want the KMS state to be constant under your dynamics.

12 May 30, 2016

Recap: We dealt with the dynamics, using a map $t \to \tau^t \in \text{Aut}(\mathcal{A})$. This map had some continuity property. We saw Hille-Yosida, which roughly said $\tau^t = e^{t\delta}$ on the domain of δ....see above.

It is not hard to show that

$$\tau^{i\gamma+s}(A) = \tau^{i\gamma}(\tau^s(A))$$

both sides are analytic, and agree on $i\mathbb{R}$, so by analyticity, they must be equal. However, if we do

$$\tau^{i\gamma+s}(A) = \tau^s(\tau^{i\gamma}(A)$$

then we don't get that nice property.

12.1 Representations

Definition 25. A representation of a $\mathcal{C}^*$-algebra $\mathcal{A}$ on a Hilbert space h, *is a* $*$*-algebra morphism*

$$\pi : \mathcal{A} \to \mathcal{B}(h).$$

If π is injective, we say the representation is faithful.

Definition 26. *Let h be a Hilbert space, $\mathcal{M} \subset \mathcal{B}(h)$. A vector $\Omega \in h$ is called* cycle *(for $\mathcal{M}$) if $\mathcal{M}\Omega$ is dense in h.*

Theorem 13 (GNS)**.** *If you have a $\mathcal{C}^*$-algebra with a state ω, then there is a representation (h_ω, π_ω) and a cyclic (for $\pi_\omega(\mathcal{A})$) vector $\Omega_\omega \in h_\omega$ such that*

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)_{h_\omega}.$$

This is unique up to unitary equivalence.

Proof. Consider $\mathcal{A}, \omega$. Define $I_\omega = \{A \in \mathcal{A} : \omega(A^*A) = 0\}$. We claim that I_ω is an ideal. Define

$$\tilde{h}_\omega = \mathcal{A}/I_\omega.$$

Define $([A], [B]) = \omega(A^*B)$ This gives us an inner product, but we still need to show that it is complete. We let $h_\omega = \overline{\tilde{h}_\omega}$. Now,

$$\pi_\omega : \mathcal{A} \to \mathcal{B}(h_\omega)$$

$$A \mapsto \pi_\omega(A)$$

$\pi_\omega(A)[B] = [AB]$, which we can extend by continuity. We need to show that $\pi_\omega(A)$ is bounded.

$$\begin{aligned}
\|\pi_\omega(A)[B]\|^2_{h_\omega} &= ([AB], [AB]) \\
&= \omega((AB)^*AB) \\
&= \omega(B^*A^*AB) \\
&\leq \|A\|^2_{\mathcal{A}}\omega(B^*B) \\
&= \|A\|^2_{\mathcal{A}}\|[B]\|^2_{h_\omega}
\end{aligned}$$

Note: I probably followed all the above poorly.

Now we defined $\Omega_\omega = [I] \in h_\omega$.

$$\pi_\omega(A)[I] = [A], \quad \pi_\omega(\mathcal{A})[I] = \pi_\omega(\mathcal{A})$$

is dense in h_ω, so Ω_ω is cyclic.

Finally, $(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = ([I],[A]) = \omega(A)$.

$(h_\omega, \pi_\omega, \Omega_\omega)$ is called the *canonical GNS representation*.

For uniqueness, suppose $(h_\omega, \pi_\omega, \Omega_\omega)$ and $(h'_\omega, \pi'_\omega, \Omega'_\omega)$ are both GNS reps. Then, they must be equivalent up to some unitary

$$\tilde{U} : D(\tilde{U}) \subset h_\omega \to h'_\omega$$

$$\pi_\omega(A)\Omega_\omega \mapsto \pi_\omega(A)\Omega'_\omega$$

$$\|\tilde{U}(\pi_\omega(A)\Omega_\omega)\|^2_{h'_\omega} = \|\pi'_\omega(A)\Omega'\|^2_{h'_\omega} = (\pi'_\omega(A)\Omega'_\omega, \pi_\omega(A)\Omega'_\omega)_{h'_\omega} = \omega(A^*A) = \|\pi_\omega(A)\Omega_\omega\|^2$$

Now extend $\tilde{U}$ to h_ω by BLT (bounded linear transformation), call it U.

$$\begin{aligned}
&(U\pi_\omega(A)\Omega_\omega, U\pi_\omega(B)\Omega_\omega) \\
&= (\pi'_\omega(A)\Omega'_\omega, \pi'_\omega(B)\Omega'_\omega)_{h'_\omega} \\
&= \omega(A^*B) = (\pi_\omega(A)\Omega_\omega, \pi_\omega(B)\Omega_\omega)
\end{aligned}$$

□

By the way: CHECK WHICH REPRESENTATION YOU ARE USING FOR COMPUTATIONS. Apparently, it will save you months of your life.

12.2 Structure Theorem for $\mathcal{C}^*$ algebras

Some results:

Theorem 14 (Hahn-Banach)**.** *Let X be a normed space, $Y \subset X$ a closed subspace. If $f : Y \to \mathbb{C}$ is a bounded liner operator, then there is an extension $F : X \to \mathbb{C}$ such that $\|F\|_{op} = \|f\|_{op}$.*

Lemma 7. *Let $A \in \mathcal{A}$ be arbitrary. There exists a state ω_A such that*

$$\omega(A^*A) = \|A\|^2$$

Proof. Let $\mathcal{B} = \{\alpha I + \beta A^*A : \alpha, \beta \in \mathbb{C}\}$. Define f on $\mathcal{B}$ by $f(\alpha I + \beta AA^*) = \alpha + \beta\|A\|^2$. We check that f is bounded:

$$\begin{aligned}
|\alpha + \beta\|A\|^2| = |\alpha + \beta\|A^*A\|| &\le \sup\{|\alpha + \beta\lambda| : \lambda \in \sigma(AA^*)\} \\
&= \sup\{|\lambda| : \lambda \in \sigma(I\alpha + \beta AA^*)\} \\
&= \|I\alpha + \beta A^*A\|
\end{aligned}$$

$\Rightarrow \|f\| \le 1$. Now, $f(I) = 1, \|I\| = 1 \Rightarrow \|f\| = 1$.

Extend f by Hahn-Banach to a functional called ω. ω_A is then a positive linear functional, $\|\omega_A\| = 1$.

$$\omega_A(A^*A) = f(A^*A) = \|A\|^2$$

□

Consider the GNS rep $(h_{\omega_A}, \pi_{\omega_A}, \Omega_{\omega_A})$ associated to $(\mathcal{A}, \omega_A)$.

$$\|\pi_{\omega_A}(A)\| \leq \|A\| \qquad C^*\text{-morphism}$$

$$\|A\|^2 = \omega_A(A^*A) = (\pi_{\omega_A}(A)\Omega_{\omega_A}, \pi_{\omega_A}(A)\Omega_{\omega_A})_{h_{\omega_A}} = \|\pi_{\omega_A}(A)\Omega_{\omega_A}\|^2_{h_{\omega_A}} \leq \|\pi_{\omega_A}(A)\|_{\mathcal{B}(h_{\omega_A})}$$

12.2.1 Direct Sums

Let I be an index set, $\{h_\alpha\}_{\alpha\in I}, \{\pi_\alpha\}_{\alpha\in I}$,

$$\pi_\alpha : \mathcal{A} \to \mathcal{B}(h_\alpha)$$

Define $h = \oplus_{\alpha\in I} h_\alpha$ to be the set of all families $\{\psi_\alpha\}_{\alpha\in I}$ $\psi_\alpha \in h_\alpha$ such that

$$\lim_F \sum_{\alpha\in F} \|\psi_\alpha\|^2_{h_\alpha} < \infty.$$

The finite subsets of I form a directed set under inclusion. $F \subset I$, F finite, $F \mapsto \sum_{\alpha\in F} \|\psi_\alpha\|^2_{h_\alpha}$ Put an inner product on H by $(\{\phi_\alpha\}, \{\psi_\alpha\}) = \sum_{\alpha\in I}(\phi_\alpha, \psi_\alpha)_{h_\alpha} = \lim_F \sum_{\alpha\in F}(\phi_\alpha, \psi_\alpha) < \infty$.

$pi = \oplus_{\alpha\in I}\pi_\alpha$ is defined by $\pi(A) = \{\pi_\alpha(A)\}_{\alpha\in I}$. We want to show that $\pi(A) \in \mathcal{B}(h), \quad \forall A \in \mathcal{A}$.

$$\begin{aligned}
\|\pi(A)\|^2 &= \sup_{\|\psi\|_h=1} (\pi(A)\psi\pi(A)\psi)_h \\
&= \sup_{\|\psi\|_h=1} \sum_{\alpha\in I} \|\pi_\alpha(A)\psi_\alpha\|^2_{h_\alpha} \\
&\leq \sup_{\|\psi\|_h=1} \sum_{\alpha\in I} \|A\|^2\|\psi_\alpha\|^2_{h_\alpha} = \|A\|^2
\end{aligned}$$

so $\pi(A)$ is bounded.

12.2.2 Structure Theorem

Theorem 15 (Structure Theorem). *Every $\mathcal{C}^*$ algebra $\mathcal{A}$ is isomorphic to a norm-closed, self-adjoint algebra of bounded operators on a Hilbert space.*

Proof. For each state ω, let $(h_\omega, \pi_\omega, \Omega_\omega)$ be the GNS rep. Let $h = \oplus_\omega h_\omega$, $\pi = \oplus_\omega \pi_\omega$. π is a $*$-algebra morphism $\Rightarrow \|\pi(A)\| \leq \|A\|$. By lemma, $\forall A \in \mathcal{A}$, there is a state ω_A such that

$$\|\pi_{\omega_A}(A)\| = \|A\|$$

$$\|\pi(A)\| \geq \|\pi_{\omega_A}(A)\| = \|A\|$$

$\Rightarrow \pi$ is injective. $\square$

13 June 1, 2016

We will introduce Modular Theory! We will introduce a $\mathcal{C}^*$-algebra, called a von Neumann algebra.

13.1 Von Neumann Algebra

Definition 27 (Von Neumann Algebra)**.** *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. The* commutant *of $\mathcal{M}, \mathcal{M}' := \{A \in \mathcal{B}(\mathcal{H}) : [A, B] = 0 \quad \forall B \in \mathcal{M}\}$. We say that $\mathcal{M}$ is a* Von Neumann Algebra *if $\mathcal{M}'' = \mathcal{M}$.*

Example 2. *If $\dim \mathcal{H} < \infty$, $\mathcal{M} = \mathcal{B}(\mathcal{H}), \quad \mathcal{M}' = \mathbb{C}I, \quad \mathcal{M}'' = \mathcal{B}(\mathcal{H})$.*

Example 3. *: If $\mathcal{M} = \mathbb{C}I, \quad \mathcal{M}' = \mathcal{B}(\mathcal{H}), \quad \mathcal{M}'' = \mathbb{C}I$*

We need another topology!

Definition 28. *The* weak operator topology (WOT) *is the smallest topology on $\mathcal{B}(\mathcal{H})$ such that*

$$A \mapsto \langle x, Ay \rangle$$

is continuous for all $x, y \in \mathcal{H}$.

Remark 3. *If $\mathcal{M}$ is a $\mathcal{C}^*$-algebra, it would be closed in the operator norm topology, because*

$$|\langle x, Ay \rangle| \leq \|x\|_{\mathcal{H}} \|Ay\|_{\mathcal{H}} \leq \|x\|_{\mathcal{H}} \|y\|_{\mathcal{H}} \|A\|_{op}$$

If $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a $\mathcal{C}^*$-algebra, it must be complete in $\|\cdot\|_{op}$. $\mathcal{B}(\mathcal{H})$ is complete with respect to the $\|\cdot\|_{op}$ norm. If $\{A_n\} \subset \mathcal{M}$ is a Cauchy sequence, then $A_n \to A \in \mathcal{B}(\mathcal{H})$. If $\mathcal{M}$ is a $\mathcal{C}^*$-algebra, we must have as well that $A \in \mathcal{M}$.

Given $(\mathcal{A}, \omega)$ we have a Hilbert space h_ω and we have $\pi_\omega(\mathcal{A}) \subset \mathcal{B}(h_\omega)$.

Proposition 24. *$\mathcal{M} \subset \mathcal{B}(\mathcal{H}), \quad \mathcal{M}^{(4)} = \mathcal{M}''$*

Proof. Let $A \in \mathcal{M}, [A, B] = 0 \forall B \in \mathcal{M}' \Rightarrow A \in \mathcal{M}''. \Rightarrow \mathcal{M} \subset \mathcal{M}''$. Now, $(\mathcal{M}'')' \subset \mathcal{M}'$ and $(\mathcal{M}') \subset (\mathcal{M}')'' \Rightarrow \mathcal{M}' = \mathcal{M}^{(3)}$. □

Definition 29. *$\mathcal{M}''$ is called the* enveloping VNA of *$\mathcal{M}$. $\mathcal{M} \subset \mathcal{M}''$, and $(\mathcal{M}'')'' = \mathcal{M}''$.*

Remark 4. *In our setting, we use $\pi_\omega(\mathcal{A})''$. Since Ω_ω is dense in h_ω, it is also dense in $\pi_\omega(\mathcal{A})''$.*

13.2 Modular Operators

Definition 30. *Let $\mathcal{M}$ be a VNA on $\mathcal{H}$. Let $\Omega \in \mathcal{H}$. Ω is cyclic for $\mathcal{M}$ if $\mathcal{M}\Omega$ is dense in $\mathcal{H}$. Ω is* separating *if $A\Omega = 0 \Rightarrow A = 0 \quad \forall A \in \mathcal{M}$*

Our construction: Consider $(\mathcal{M}, \Omega)$., where Ω is cyclic and separating. This operator:

$$\Delta_\Omega : D(\Delta) \to \mathcal{H}$$

will be important. So we'll construct it. It's Jane favorite operator.

Proposition 25. *Ω is cylic for $\mathcal{M} \iff \Omega$ is separating for $\mathcal{M}'$.*

Proof. Suppose Ω is cyclic for $\mathcal{M}$. Let $A' \in \mathcal{M}'$ such that $A'\Omega = 0$, $\forall B \in \mathcal{M}$. Then $A'B\Omega = BA'\Omega = 0, \Rightarrow A'\overline{[\mathcal{M}\Omega]} = \{0\}, \Rightarrow A'\mathcal{H} = \{0\}, \Rightarrow A' = 0$.

Conversely, suppose Ω is separating for $\mathcal{M}'$. Define P', the projection onto $\overline{[\mathcal{M}\omega]}$. Our goal is to show $P' = I$.

$$(I - P')\Omega = \Omega - \Omega = 0.$$

Let $\xi \in \mathcal{H}$ be arbitrary. $\xi = \xi_0 + \lim_{n\to\infty} A_n\Omega$, $A_n \in \mathcal{M}$. Let $B \in \mathcal{M}$,

$$BP'\xi = B \lim_{n\to\infty} A_n\Omega = \lim_{n\to\infty} BA_n\Omega = P'B\xi_0 + P'B \lim_{n\to\infty} A_n\Omega = P'B\xi.$$

$$\Rightarrow I - P' \in \mathcal{M}' \Rightarrow I - P' = 0 \Rightarrow P' = I.$$

and hence $\bar{[}\mathcal{M}\Omega] = \mathcal{H}$.

Remark 5. *If Ω is cyclic and separating for a VNA $\mathcal{M}$, then Ω is cyclic and separating for $\mathcal{M}'$. This gives us two ways to generate a dense set in $\mathcal{H}$, using the same vector but different operators.*

□

13.3 The Star Operators

They're called that because they're great and implement the adjoints. Sort of.

Definition 31. *For each tuple $(\mathcal{M}, \Omega)$, we define a* star operator*:*

$$S_0^{\mathcal{M},\Omega} A\Omega = A^*\Omega, \qquad A \in \mathcal{M}.$$

$$F_0 A'\Omega = A'^*\Omega, \qquad A' \in \mathcal{M}'.$$

If Ω is cyclic and separating for $\mathcal{M}$ then $D(S_0)$, $D(F_0)$ are dense.

$$A\Omega = B\Omega \Rightarrow (A - B)\Omega = 0 \Rightarrow A = B$$

because Ω is separating. so S_0, F_0 are well-defined, they are maps on the Hilbert space that are not necessarily in $\mathcal{B}(\mathcal{H})$, such that $A\Omega \mapsto A^*\Omega$.

Proposition 26. *S_0 and F_0 are closable. Also, $\bar{S}_0 = F_0^*$, $\bar{F}_0 = S_0^*$.*

Side remark about adjoints of unbounded operators: let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$.

$$D(A^*) := \{\phi \in \mathcal{H} : \psi_\phi.(A\varphi, \phi) = (\varphi, \psi_\phi)\forall\varphi \in \mathcal{H}\}$$

then we define $A^*\phi := \psi_\phi$. Note that this is necessary because of the unboundedness.

Proposition 27. *A is closable $\iff$ $D(A^*)$ is dense.*

In this case, $\bar{A} = A^{**}$.

Proof. Let $A \in \mathcal{M}$, $A' \in \mathcal{M}'$. Then,

$$\begin{aligned}(A'\Omega, S_0 A\Omega) &= (A'\Omega, A^*\Omega) \\ &= (\Omega, A'^* A^*\Omega) \\ &= (\Omega, A^* A'^*\Omega) \\ &= (A\Omega, A'^*\Omega) \\ &= (A\Omega, F_0 A'\Omega)\end{aligned}$$

Therefore, we have

$$\underbrace{(A'\Omega, S_0 A\Omega)}_{(A\Omega, S_0^* A'\Omega)} = (A\Omega, F_0 A'\Omega)$$

we can define $S_0^* A\Omega := F_0 A'\Omega$, which implies $D(F_0) \subset D(S_0^*)$, which implies $D(S_0^*)$ is dense, so S_0 is closable. A similar argument shows that F_0 is closable.

We want to show $\bar{F}_0 = S_0^*$ (use the fact that since F_0 is closable, $\bar{F}_0 = F_0^{**}$).

Let $\xi \in D(S_0^*)$. Set $\psi = S_0^*\xi$, and take $A \in \mathcal{M}$.

$$(A\Omega, \psi) = (A\Omega, S_0^*\xi) = (\xi, S_0 A\Omega) = (\xi, A^*\Omega).$$

...we will get back to this. □

Since $S = \bar{S}_0 = F_0^*$ is closed, we can take the polar decomposition.

$$S = \underbrace{J}_{\text{anti-unitary}} \underbrace{\Delta^{1/2}}_{\text{self-adjoint}}$$

this composition is unique.

Definition 32. Δ *is the* modular operator*,* J *is the* modular conjugation*.*

There are many nice identities involving S, F, J, Δ which you can prove algebraically.

Example 4. $J^2 = I, J = J^*, J\Delta^{1/2}J = \Delta^{-1/2}$

It is doable to prove these identities yourself. Detailed proofs are available in Jane's thesis.

Theorem 16 (Tomita-Takesaki)**.**

$$J\mathcal{M}J = \mathcal{M}', \qquad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$$

Note that the first is really useful, because $J^2 = I$, so we may insert these J's everywhere. The second is useful because it gives us an automorphism on $\mathcal{M}$, and a concrete map $t \mapsto \Delta^{it} \cdot \Delta^{-it}$.

14 June 2, 2016

Recall that: $\mathcal{M}' = \mathcal{M}'''$ and $\mathcal{M}'' = \mathcal{M}^{(4)}$ and $\mathcal{M} \subset \mathcal{M}''$. For any subalgebra, $\mathcal{M}, \mathcal{M}''$ is a VNA. $\mathcal{M}$ is a VNA $\iff$ is closed in WOT. As a result, VNAs are $\mathcal{C}^*$-algebra...etc. Repeated last class' theorems.

Proposition 28. *S_0 and F_0 are closable. Define $S = \bar{S_0}$, $F = \bar{F_0}$. $\bar{S_0} = F^*$, $\bar{F_0} = S^*$.*

Reminder: $\bar{S_0} = S_0^{**}$, if $D(S_0^*)$ is dense.

We took the unique polar decomposition of the anti-linear operator S.

$$S = J\Delta^{1/2}$$

Where, J is anti-unitary, and $\Delta^{1/2}$ is positive. One can verify that $\Delta^{1/2} = \sqrt{(\Delta^{1/2})^2}$, where the square is the composition of the operator with itself, and the square root is defined using spectral theory.

Some useful identities include:

$$FS = \Delta, \quad SF = \Delta^{-1}, \quad J^2 = I, \quad J = J^*, \quad J\Delta^{1/2}J = \Delta^{-1/2}.$$

Let $(\mathcal{A}, \tau, \omega)$ be a $\mathcal{C}^*$-dynamical system. The GNS rep shows that

$$(h_\omega, \pi_\omega : \mathcal{A} \to \mathcal{B}(h_\omega), \Omega_\omega)$$

such that $\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$.

$\pi_\omega(\mathcal{A})$ is not necessarily a VNA, but $\pi_\omega(\mathcal{A})''$ is (stabilization of the '). Let $\mathcal{M} = \pi''_\omega(\mathcal{A})$. The GNS rep gives Ω is cyclic for $\pi_\omega(A)$, and hence Ω is cyclic for $\mathcal{M}$. In general, Ω is not separating for $\mathcal{M}$.

Proposition 29. *If ω is a $\beta - \tau$-KMS state for* any *dynamics τ, then Ω is separating.*

Ingredients for proof:

1. (Reformulation of KMS condition) ω KMS state. Let $D_\beta = \{z \in \mathbb{C} : 0 < \text{Im } z < \beta\}$, $\quad \forall A, B \in \mathcal{A} \quad \exists F_{A,B} : \bar{D_\beta} \to \mathbb{C}$ such that $F_{A,B}(t) = \omega(A\tau^t(B)) \quad \forall t \in \mathbb{R}$ and $F_{A,B}$ is analytic on D_β and bounded and continuous on the closure.

2. (Edge of the wedge) Let $\mathcal{O}$ be an open set in $\mathbb{C}$. If $f : \mathcal{O} \to \mathbb{C}$ such that $f\big|_{\mathcal{O}\cap H^+}$ is holomorphic and $f(t) = 0$ for $t \in \mathbb{R}$. Then $f\big|_{\mathcal{O}\cap H^+}$ is constantly 0.

3. (Extending dynamics and states). We can extend ω such that $\hat{\omega}(B) = (\Omega_\omega, B\Omega_\omega), B \in \mathcal{M}$, in the sense that $\hat{\omega}(\pi_\omega(A)) = \omega(A), A \in \mathcal{A}$. We may also extend τ, following the proof of the uniqueness of the GNS rep.

If $(h_\omega, \pi_\omega, \Omega_\omega)$ and $(h'_\omega, \pi'_\omega, \Omega'_\omega)$ are two GNS reps for $(\mathcal{A}, \omega)$ then

$$\exists U : h_\omega \to h'_\omega$$

$$\pi_\omega(A)\Omega_\omega \mapsto \pi_\omega(A)'\Omega'_\omega$$

so the representation is unique to a unitary. *Note that this unitary is may not be unique!.* Suppose ω is τ-invariant (e.g. KMS state). $(h_\omega, \pi_\omega, \Omega_\omega), (h_\omega, \pi_\omega \circ \tau^t, \Omega_\omega)$ are both GNS reps for ω.

$$(\Omega_\omega, \pi_\omega \tau^t(A), \Omega_\omega) = \omega(\tau^t(A)) = \omega(A).$$

Then, there is a map $t \mapsto U(t)$, where $U(t) : \pi_\omega(A) \mapsto \pi_\omega \circ \tau^t(A)\Omega_\omega$ is continuous (wrt some topology) and

$$U(t)\Omega_\omega = \Omega_\omega, \quad \forall t.$$

Define $\hat{\tau}^t(B) = U(t)BU(t)^*$. This is a continuous one-parameter automorphism group.

$\hat{\omega}$ is a $\beta - \hat{\tau}$ KMS state.

$$\hat{\omega}(\pi_\omega(A\tau^{i\beta}(B))) = \omega(A\tau^{i\beta}(B)) = \omega(BA) = \hat{\omega}(\pi_\omega(BA))$$

Lemma 8. *Let $\mathcal{M}$ be a $\mathcal{C}^*$-algebra on h. Ω is cyclic and we induce a state $\omega(A) = (\Omega, A\Omega), A \in \mathcal{M}$. Then if $\omega(A^*A) = 0 \Rightarrow \omega(AA^*) = 0, \quad \forall A \in \mathcal{M}$, the vector Ω is separating.*

Proof. Suppose $A\Omega = 0$. Then $BA\Omega = 0$ for all $B \in \mathcal{M}$.

$$\Rightarrow (\Omega, (BA)^*(BA)\Omega) = \omega((BA)^*(BA)) = 0$$

$$\Rightarrow \omega((BA)(BA)^*) = 0 \Rightarrow (\Omega, (BA)(BA)^*\Omega) = 0 \Rightarrow \|(BA)^*\Omega\| = 0.$$

$$\Rightarrow (BA)^*\Omega = 0, \quad \Rightarrow A^*B^*\Omega = 0 \quad \forall B \in \mathcal{M}$$

$$\Rightarrow A^* = 0$$

$$\Rightarrow A = 0.$$

□

Proof of main result. Let $\hat{\omega}$ be $\hat{\tau} - \beta$-KMS. Let $A \in \mathcal{M}$ be such that $\hat{\omega}(A^*A) = 0$. We want to show that $\hat{\omega}(AA^*) = 0$. Let $F_{A^*,A}(z)$ be the "KMS function". Then,

$$\forall t \in \mathbb{R}, \qquad \begin{aligned} F_{A^+,A}(t) &= \hat{\omega}(A^*\hat{\tau}^t(A)) = (\Omega_\omega, A^*U(t)AU(t)^*\Omega_\omega) \\ &= (A\Omega_\omega, U(t)A\Omega_\omega) \\ &= 0 \end{aligned}$$

since $A\Omega_\omega = 0$. Indeed, $\omega(A^*A) = 0$, so $(\Omega_\omega A^*A\Omega_\omega) = 0$, so $\|A\Omega_\omega\| = 0, \Rightarrow A\Omega_\omega = 0$.

By the edge of the wedge, $F_{A^*,A}(z) = 0 \quad \forall z \in D_\beta$. Finally, $\hat{\omega}(AA^*) = F_{A^*,A}(i\beta) = 0$. By the lemma, Ω_ω is separating.

We had $\mathcal{A}, \omega$ a KMS state for the dynamics τ, which we mapped to $\mathcal{M} = \pi_\omega(\mathcal{A})'', \Omega_\omega$ which is cyclic and separating. We can define the associated modular operators

$$J_\omega, \Delta_\omega$$

$$\sigma_\omega^t(A) = \Delta^{-it/\beta}A\Delta^{it/\beta}, \qquad \forall A \in \mathcal{M}$$

By Tomita-Takasaki, $\sigma_\omega^t \in \text{Aut }(\omega)$, $\tau \mapsto \sigma_\omega^t$ has some nice continuity properties. σ_ω^t is called the modular dynamics.

$$
\begin{aligned}
\hat{\omega}(A\sigma_\omega^{i\beta}(B)) &= (\Omega, A\Delta B\Delta^{-1}\Omega) \\
&= (\Omega.A\Delta BSF\Omega) \\
&= (\Omega, A\Delta B\Omega) \\
&= (\Omega, A\Delta^{1/2}JJ\Delta^{1/2}B\Omega) \\
&= (\Delta^{1/2}A^*\Omega, JB^*\Omega) \\
&= (B^*\Omega, J\Delta^{1/2}A^*\Omega) \\
&= (B^*\Omega, A\Omega) \\
&= \hat{\omega}(BA)
\end{aligned}
$$

We will finish this proof later.

□

For the moment, consider $\mathcal{A} = \mathcal{B}(\mathcal{H})$, ω a trace state. $\omega(A) = \mathrm{tr}(\rho_\omega A)$. Let's see an example of the GNS construction for this special case, and how the modular operators act. Let $\pi_\omega(A) = [A], \Omega_\omega = [I]$. $\mathcal{I} = \{A : \omega(A^*A) = 0\}$. Then

$$S : [AI] \mapsto [A^*I]$$

$$
\begin{aligned}
J : [A] &\mapsto [\rho_\omega^{1/2} A^* \rho_\omega^{1/2}] \\
\Delta^{1/2} : [A] &\mapsto [\rho_\omega^{1/2} A \rho_\omega^{-1/2}]
\end{aligned}
$$

then

$$
\begin{aligned}
&(J\pi_\omega(A)\Omega, J\pi_\omega(B)\Omega) \\
&= ([\rho_\omega^{1/2} A \rho_\omega^{-1/2}], [\rho_\omega^{1/2} B \rho_\omega^{-1/2}]) \\
&= \mathrm{tr}(\rho_\omega \rho_\omega^{1/2} A \rho_\omega^{1/2} \rho_\omega^{1/2} B^* \rho_\omega^{-1/2}) \\
&= \mathrm{tr}(\rho_\omega B^* A) \\
&= (\pi_\omega(B)\Omega, \pi_\omega(A)\Omega) \\
&\Rightarrow \text{ J is anti-unitary} \\
&(\Delta^{1/2}\pi_\omega(A)\Omega, \pi_\omega(B)\Omega) \\
&= ([\rho_\omega^{1/2} A \rho_\omega^{-1/2}], [B]) \\
&= \mathrm{tr}(\rho_\omega \rho_\omega^{-1/2} A^* \rho_\omega^{1/2} B) \\
&= \mathrm{tr}(\rho_\omega A^* \rho_\omega^{1/2} B \rho_\omega^{-1/2}) \\
&= ([A], [\rho_\omega^{1/2} B \rho_\omega^{1/2}) \\
&= (\pi_\omega(A)\Omega, \Delta^{1/2}\pi_\omega(B)\Omega) \\
&\Rightarrow \Delta^{1/2} \text{ self-adjoint} \\
&J\Delta^{1/2}\pi_\omega(A)\Omega \\
&= J\Delta^{1/2}[A] \\
&= J[\rho_\omega^{1/2} A \rho_\omega^{-1/2}] \\
&= [\rho_\omega^{1/2}(\rho_\omega^{-1/2} A^* \rho_\omega^{1/2})\rho_\omega^{-1/2}] \\
&= [A^*] \\
&= \pi_\omega(A)^*\Omega.
\end{aligned}
$$