Stability & invariant random subgroups

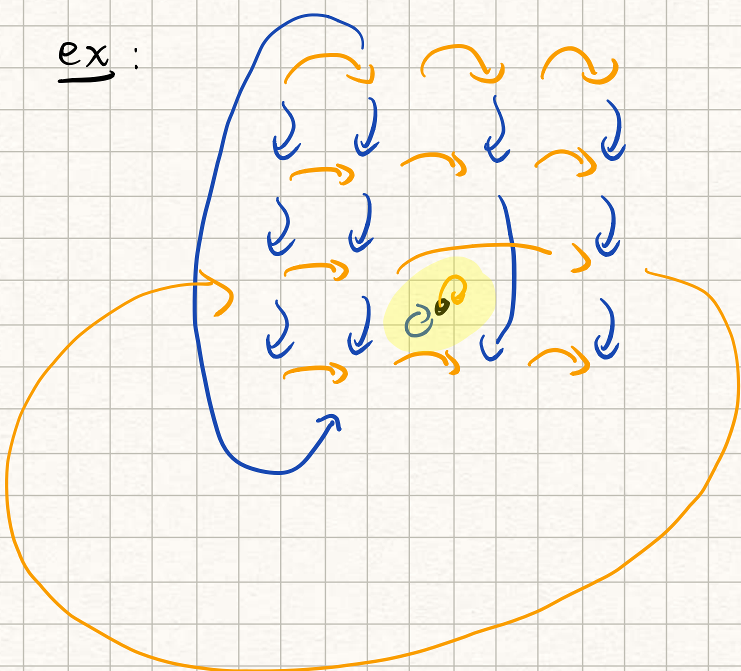
Motivating problem

Are almost commuting perm close to commuting perm?

S_n n large.

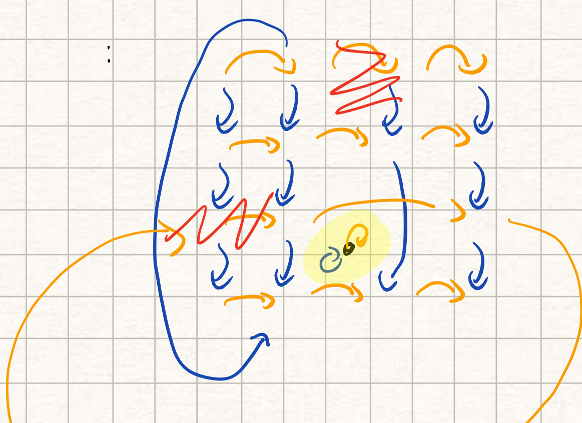
σ, τ

ex:



commuting almost everywhere

how do you make it commuting?



→ Sort of have to change n things on n^2 arrows. ~ small when n large.

Def: $\sigma, \tau \in \mathrm{Sym}(n)$

$$d(\sigma,\tau) = \frac{1}{n} \left| \{ i \in n \mid \sigma(i) \neq \tau(i) \} \right|$$

$$\in [0,1]$$

↑ normalized for all n.

Thm: (AP) $\forall \varepsilon > 0, \exists \delta > 0 : \forall n \in \mathbb{N}$

$\forall \sigma, \tau \in \mathrm{Sym}(n)$

$$d(\sigma\tau, \tau\sigma) < \delta \Rightarrow \exists \sigma', \tau' \in \mathrm{Sym}(n)$$

st $d(\sigma, \sigma') < \varepsilon$, $d(\tau, \tau') < \varepsilon$

$\sigma'\tau' = \tau'\sigma'$!

Invariant random subgroup

Let Γ be a grp,

$$\mathrm{Sub}(\Gamma) := \{ H < \Gamma \mid H \text{ subgrp} \}$$

$$\subseteq 2^\Gamma$$

closed subset

↑ compact by Tychonoff

Γ acts on $\mathrm{Sub}(\Gamma)$ by $g \cdot H = gHg^{-1}$

<u>Fixpts</u>: $\Gamma \circlearrowright \mathrm{Sub}(\Gamma) = N\mathrm{Sub}(\Gamma)$

$= \{ H \trianglelefteq \Gamma \mid \text{normal subgrp} \}$

$$\mathrm{Sub}(\Gamma) \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{transitive action on} \\ \text{sets w/ bpt.} \end{array} \right\} \Big/ \sim$$

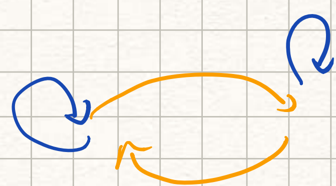
$$H \longmapsto \Gamma \circlearrowright (\Gamma/H, H = \text{bpt})$$

$$\mathrm{Stab}(x) \longleftarrow (\Gamma \circlearrowright X, x \in X)$$

$\rightsquigarrow \Gamma = F_n$, $n \geq 2$

$\mathrm{Sub}(F_n) \longleftrightarrow$ Schreier graph (pointed) on n gens.

ex: $n = 2$



$\cup|$

$N\mathrm{Sub}(F_n) \longleftrightarrow$ Cayley graph of corresponding quotient (n gens)

$=$ space of marked grps

$$\{(\Lambda, x_1 \dots x_n) \mid \underset{x_i \in \Lambda}{\langle x_1 \dots x_n \rangle} = \Lambda\}\Big/_{\sim}$$

Q: What does it mean that $\lim_{n \to \infty} H_n = H$ in $\mathrm{Sub}(\Gamma)$ or $\mathrm{NSub}(\Gamma)$? (assuming conv.)

$$g \in H \iff \exists k \text{ st } \forall n \geq k,\ g \in H_n$$

$$H = \bigcup_k \bigcap_{n \geq k} H_n$$

Q: How can we approximate groups (in the space of marked grps by finite grps OR normal subgrps of F_n by finite index subgrps) ?

A: No. Finite grps are NOT dense.

→ fp simple groups is NOT a limit of fin. grps in the space of marked grps.

weak $*$-topology

<u>Denote</u>: by PSub(Γ) the space of Borel prob. meas on Sub(Γ).

The action of Γ on Sub(Γ) gives rise to an action of Γ PSub(Γ).

IRS(Γ) = fixed pt of $\Gamma \curvearrowright PSub(\Gamma)$

invariant rand. subgrp

<u>examples</u>:

1) $NSub(\Gamma) \subseteq IRS(\Gamma)$

$$H \longmapsto \delta_H$$

2) Let $H < G$ f.i.

Take equi. distribution on the orbit

$$\{ gHg^{-1} \mid g \in \Gamma \}$$

3) $\Gamma \curvearrowright (X, \mu)$ prob. meas. space

($\Gamma \curvearrowright$ by Borel auto which preserves μ)

$$Stab : X \longrightarrow Sub(\Gamma)$$
$$x \longmapsto Stab(x)$$

$$Stab_*(\mu) \in PSub(\Gamma)$$

Q. (Gromov) : Is it true that for every normal subgrp $N \triangleleft F_n$, δ_N is a limit of finite index IRS ?

Q'; Same Q in the picture using Schreier graph: Is the Cayley graph of F_n/N a "limit" of Schreier graph? $\left(G_R = F_n/H_R, x_1, \dots, x_n \right)$

Here, lim means that for $\forall R > 0$ (radius), the probability of a randomly chosen R-ball looks like in $Cay(F_n/N, x_1, \dots, x_n)$ tends to 1
is isomorphic

NOTE: F_n/N is called sofic if it is such a limit.