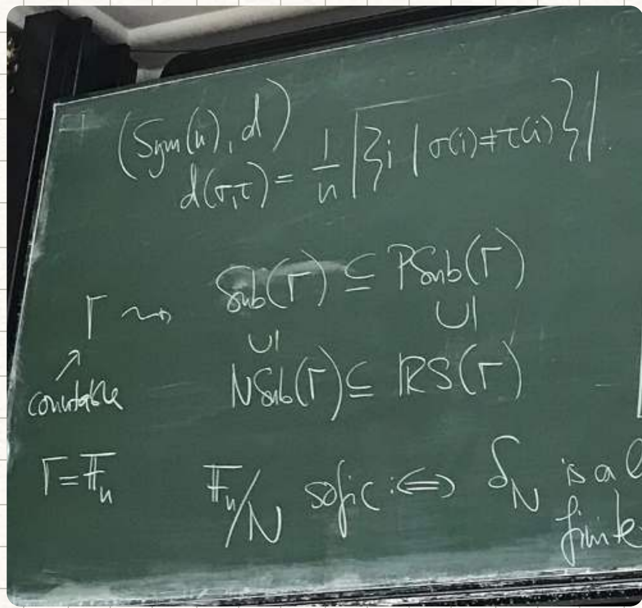
last time →



<u>**Remark**</u>: $\Gamma = \mathbb{F}_n$,

$\mathbb{F}_n/N$ sofic $\Leftrightarrow$ δ_N is a limit of finite index IRS in $\mathrm{IRS}(\mathbb{F}_n)$.

Def: We say for $(\mu_n)_{n\in\mathbb{N}}$, $\mu_n \in \mathrm{PSub}(\Gamma)$,

$\mu_n \to \mu$ weak-*

$\Leftrightarrow$ $\forall f \in C(\mathrm{Sub}(\Gamma))$ $\int f\, d\mu_n \to \int f\, d\mu$.

On $\mathrm{Sub}(\Gamma)$, H is clopen so δ_H is continuous.

useful criterion since many function on $\mathrm{Sub}(\Gamma)$ are continuous,

ex: $F, F' \subseteq \Gamma$ finite

$$X_{F,F'} = \left\{ H < \Gamma \;\middle|\; \begin{array}{l} F \subseteq H \\ F' \cap H = \emptyset \end{array} \right\}$$

clear fact.

$\to$ $\chi_{X_{F,F'}} : \mathrm{Sub}(\Gamma) \longrightarrow \{0,1\}$

continuous.

$\mathrm{Sub}(\mathbb{F}_n) \xleftrightarrow{1-1}$ pointed Schreier graphs of $\mathbb{F}_n$.

$N \longrightarrow \mathbb{F}_n \curvearrowright \mathbb{F}_n/N$

Thm:

S_N is a limit of finite index IRS (G_m)

$\iff \exists$ sequence of finite pointed Sch. graph $\vee$ of $\mathbb{F}_n$ st

$\forall R \geq 0$, $\mathbb{P}\left(\begin{array}{l}\text{random } R\text{-ball in } \cancel{\text{the}} \\ \text{Schreier graph is labelled} \\ \text{isomorphic to the } R\text{-ball} \\ \text{in } \mathrm{Cay}(F_n/N, x_1 \ldots x_n)\end{array}\right)$

$\longrightarrow 1$ as $m \to \infty$

This gives a finite set of permutations

$\sigma_1, \ldots, \sigma_n \in \mathrm{Sym}(V(G_m))$

st $\forall w \in N$,

$d(w(\sigma_1, \ldots, \sigma_n), \mathbb{1}_{V(G_m)}) \longrightarrow 0$ (as $m \to \infty$)

$\forall w \notin N$

$d(w(\sigma_1, \ldots \sigma_n), \mathbb{1}_{V(G_m)}) \to 1$

Alt Def Let Γ be a grp, fg,

Γ is said to be sofic if $\forall F \in \Gamma$ finite,

$\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$,

$\exists \varphi : \Gamma \to \mathrm{Sym}(n)$ (Not necessarily hom) map

st

1) $d(\varphi(gh), \varphi(g)\varphi(h)) < \varepsilon \quad \forall g,h \in F$

2) $d(\varphi(g), 1_n) > 1 - \varepsilon \quad \forall g \in F \setminus \{1_\Gamma\}.$

($>1/2$ is enough)

If Γ fg, $\Gamma = \mathbb{F}_n / N$

$\pi : \mathbb{F}_n \underset{\sigma}{\rightleftarrows} \Gamma$

Metric Ultraproducts

Let $\mathcal{U}$ be non-princ. ultra-filter.

$$\prod_{n\in\mathbb{N}} \mathrm{Sym}(n) \supseteq N_{\mathcal{U}} := \left\{ (a_n)_n \;\middle|\; \lim_{n\to\mathcal{U}} d(a_n, 1_n) = 0 \right\}$$

$$(\mathrm{Sym}_{\mathcal{U}}, d) := \prod_{n\in\mathbb{N}} \mathrm{Sym}(n) \Big/ N_{\mathcal{U}}$$

$$d([\sigma_n], [\tau_n]) = \lim_{n\to\mathcal{U}} d_{\mathrm{Sym}(n)}(\sigma_n, \tau_n)$$

Another alt def (sofic)

$$\Gamma \text{ sofic} \Leftrightarrow \Gamma \hookrightarrow \mathrm{Sym}_{\mathcal{U}}$$

Could also require

$d(\varphi(g), 1_{\mathrm{Sym}_{\mathcal{U}}})$

$= 1$ if $g \neq 1_\Gamma$.

Def (Amenable grps)

Γ amenable $\iff$ $\forall \varepsilon > 0$ fg any $S = S^{-1}$

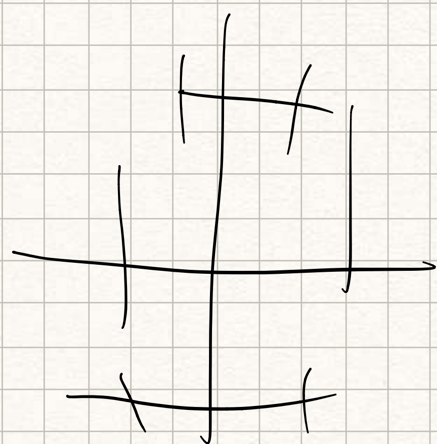
$\exists F \subseteq \Gamma$ finite st

$$|SF \cup F| < (1+\varepsilon)|F|$$

ex: $\Gamma = \mathbb{Z}$, $S = \{\pm 1\}$

$F = \{0, 1, 2, \dots, n\}$ $n \gg$ [illegible]

ex (non-amenable) F_2.



Thm (Gromov) Amenable grps are sofic.

Consider the restriction of $\Gamma \curvearrowright \Gamma$ to F and correct $g|_F$ to a permutation of F.

$\rightsquigarrow$ $\Gamma \hookrightarrow \mathrm{Sym}_{\mathcal{U}}$

Thm (Newmann)

Let $\varphi, \psi : \Gamma \to Sym_n$ hom &

Γ amenable.

TFAE

1) $d(\varphi(g), 1) = d(\psi(g), 1) \quad \forall g \in \Gamma$

2) φ, ψ conj.