Flows & foliations

In pre-geom'ation era

→ M closed, irred, connected, oriented 3-manif w/ infinite π_1, atoroidal.

$\rightsquigarrow$ hyp.

→ M^3 f co-orient, taut foliation

$\Downarrow$ (Thurston's univ. circle)

M contains a pair of essential laminations

$\Downarrow$ (Graha-Kazez)

$\pi_1(M)$ is word-hyp.

→ M. $\mathbb{Q}HS^3$ rational hom. 3-sphere

M is not L-space $\Longleftrightarrow$ (conjecture) $\pi_1(M)$ is LO

$\Uparrow$ $\Downarrow$ conj. $\qquad\qquad$ $\Downarrow$

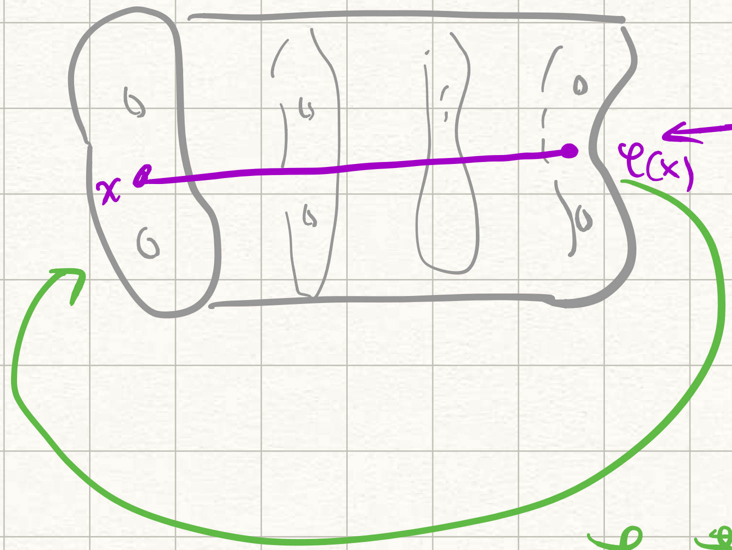
M has taut foliation $\Longrightarrow$ $\pi_1(M)$ is CO (cyclically ord.) $\Longleftrightarrow$ $\pi_1(M) \overset{+}{\curvearrowright} S^1$ + inv. lamins.

M q-geod flow

M pseudo-A flow

M has essential lamination w/ "mild assumpt" on the compl. region.

① M^3, $\mathcal{F}$ ~~co-dim 1~~ 2-dim foliation,
f is called taut if
$\exists$ closed loop in γ in M &
$\mathcal{F}$ which meets at every
leaf.



← the loop

By making mapping torus & gluing, get a taut foliation

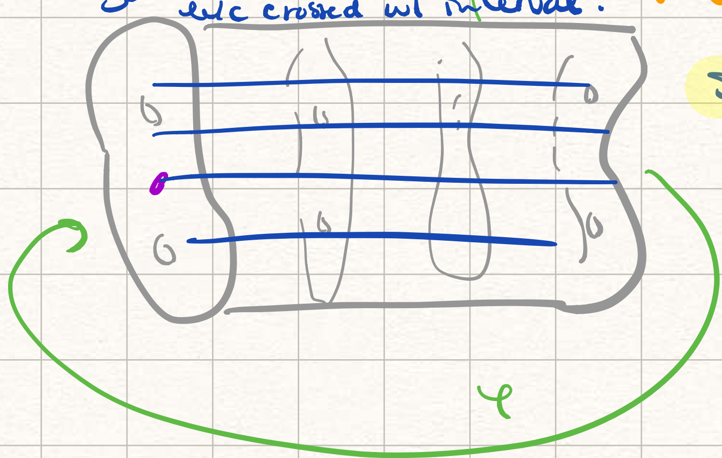
φ the glue.

Post-geometrization era

② M X flow ($\mathbb{R}$-action on M), X called q-geods if all the flow lines of $\tilde{X}$ in $\tilde{M}$ univ cover are q-geods.

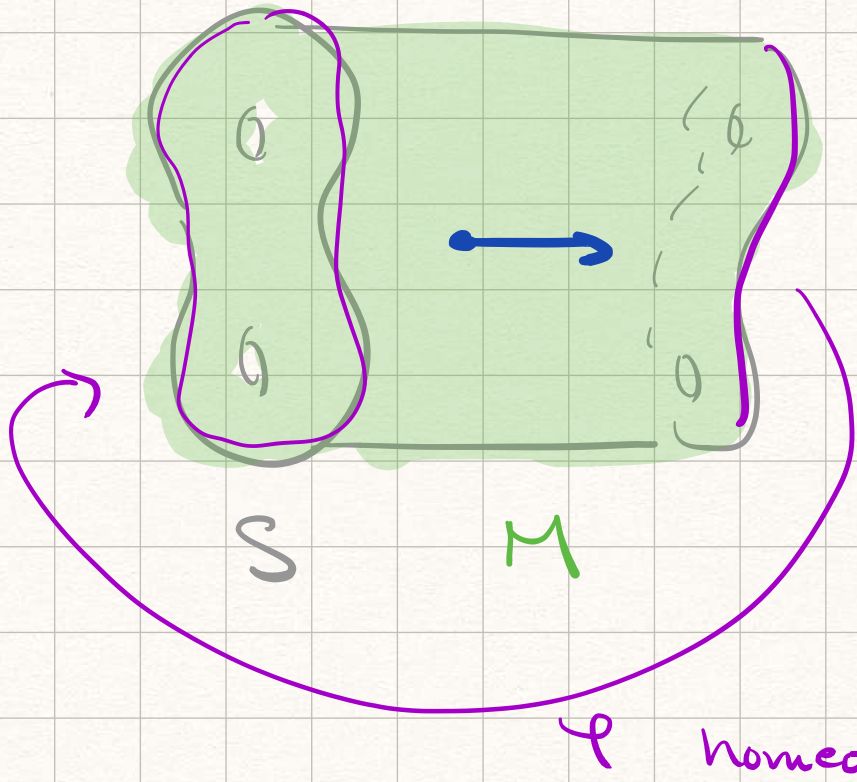
get this $\mathbb{R}$ action w/c crossed w/ interval.

→ makes q-geods in the lift.



suspension flow.

$\Sigma_g \times I$, $\{p\} \times I$, $\forall p \in \Sigma_g$ is the flow of quotient manif



$$M = \overset{x}{S} \times \overset{t}{[0,1]} / \varphi$$

flow: $\frac{\partial}{\partial t}$ on $S \times [0,1]$

$\downarrow$

X on M " $\frac{\partial}{\partial t} / \varphi$ "

vector field

flow $(X_t)_{t \in \mathbb{R}}$ grp w/ one param of diffeo on M.

$\mathbb{R}$-action : take $y \in M$,

$\mathbb{R} \times M \to M$

$t \cdot y \mapsto X_t(y) \in M$

start from y, go t time in X vector field.

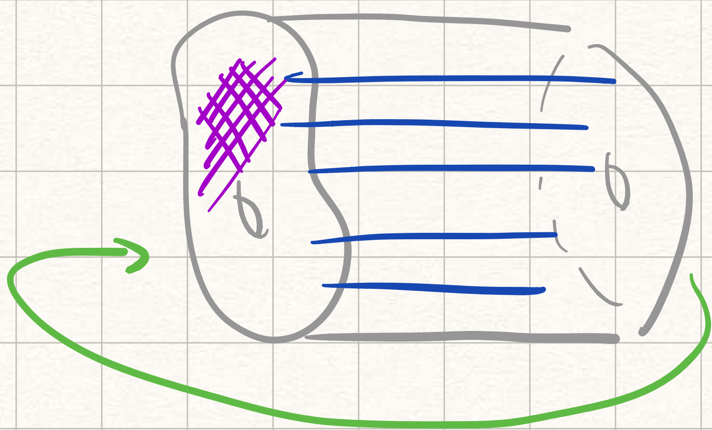
a flow

③ M, X, X said to be Anosov

if $TM = E^s \oplus E^u \oplus TX$

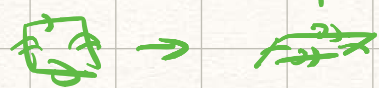
in one foliation, gets compressed (stable part)

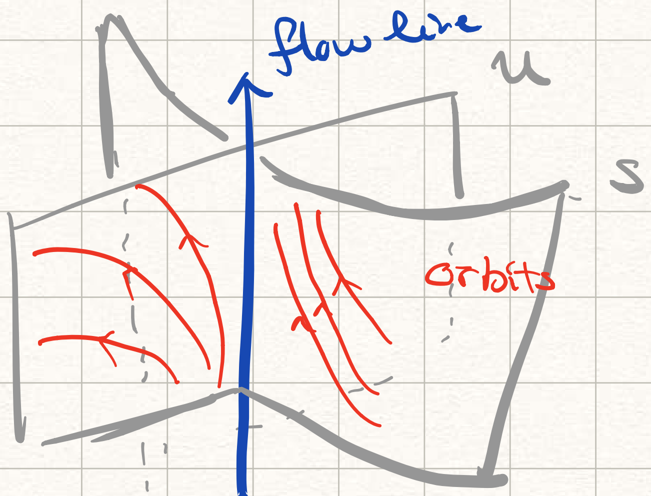
foliation gets expanded (unstable)



ℓ-Anosov (as map)

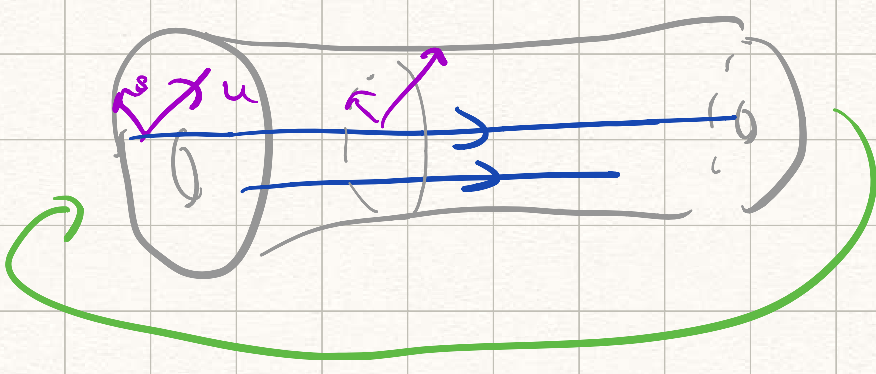
ex



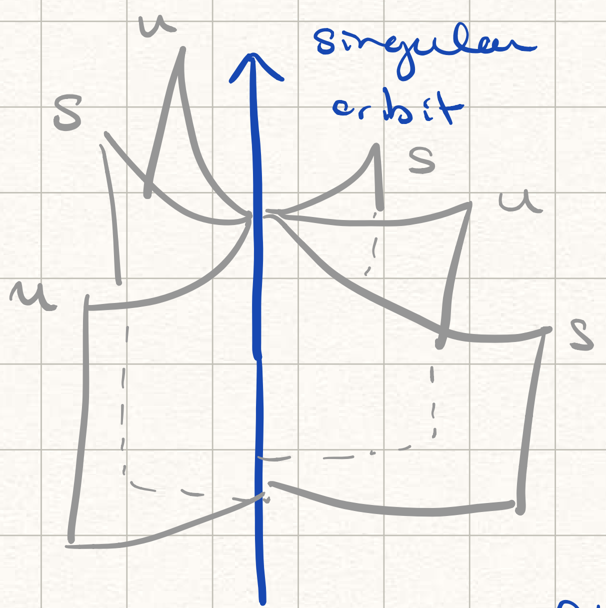


Anosov

Suspension flow from Anosov diffeo $\Rightarrow$ Anosov flow.

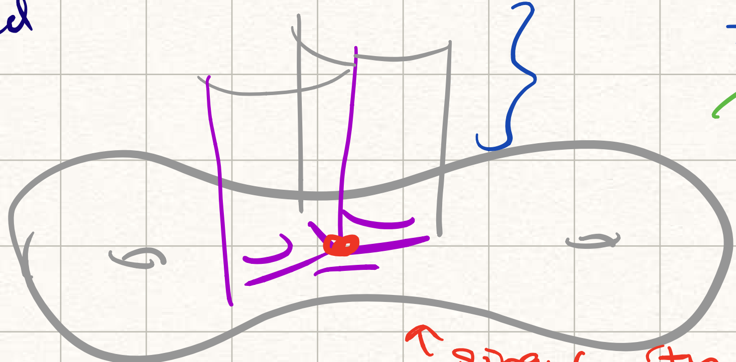


Pseudo-Anosov ($g \in MCG$ property)



& looks like this when crossed w/ interval

creates pseudo-A flow.



Pseudo Anosov (map)

↑ singularity of tangent space

the map has expanding & contracting dirs.

⓪ Tangential geom & taut foliation

Recall: The classical unif'ation thm says the simply connected Riemann surfaces are equiv. to $S^2, \mathbb{H}^2, \mathbb{E}^2$

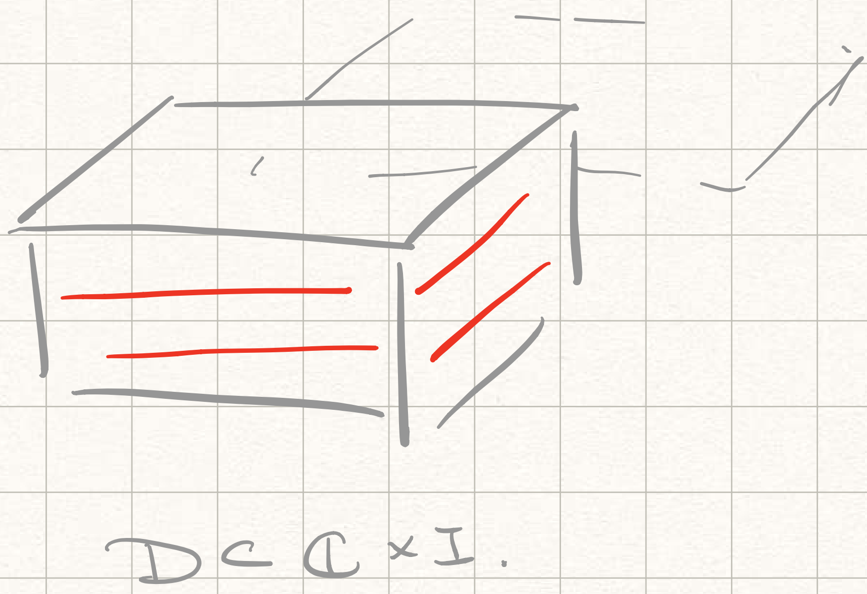
Σ compact RS

$\chi(\Sigma) < 0 \iff \tilde{\Sigma} \cong \mathbb{H}^2$

$= 0 \iff \mathbb{E}^2$

$> 0 \iff S^2$

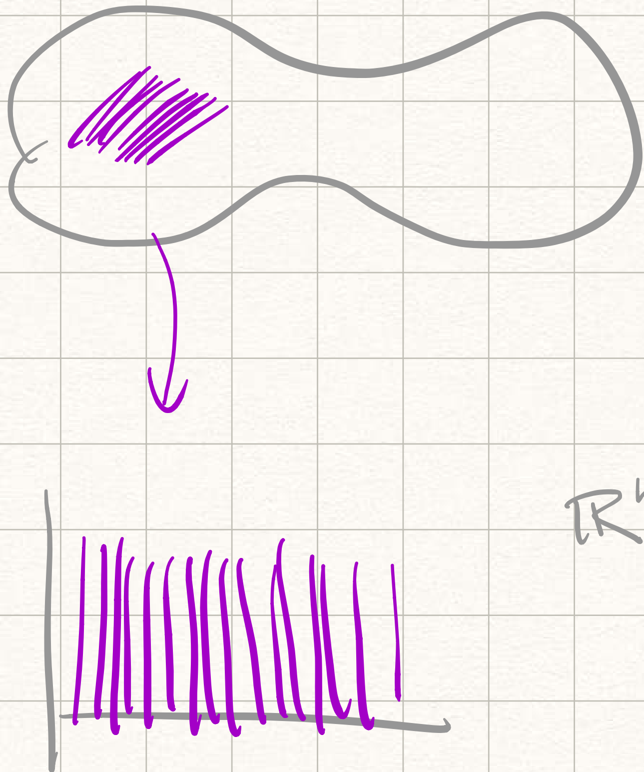
Cand's unif thm for Riemann surface lamination



lamination = "foliation on a closed space.

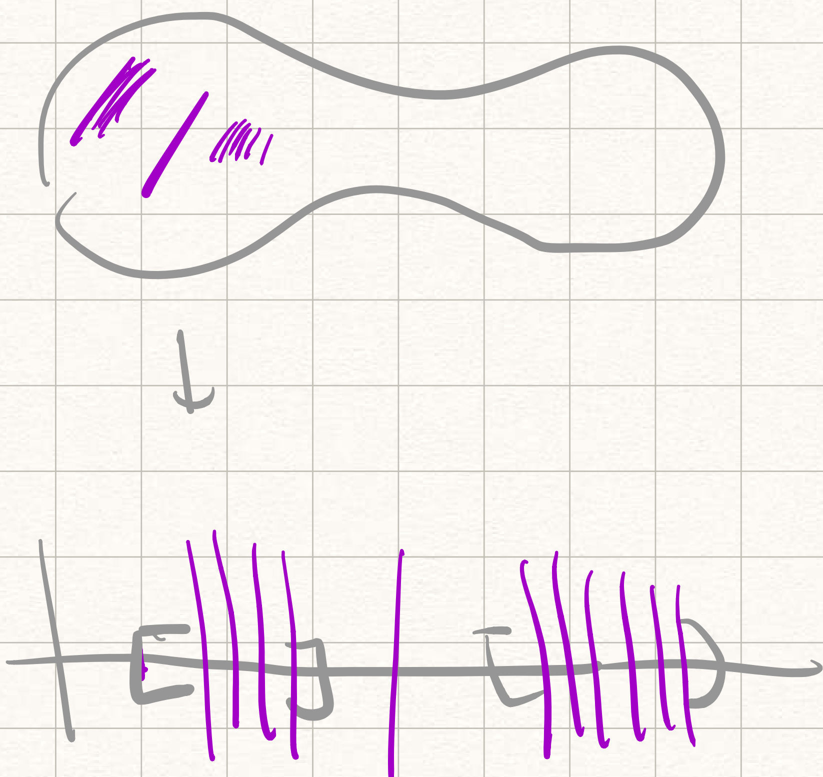
foliation $\mathcal{U} \xrightarrow{\sim} \mathbb{R}^k \times \mathbb{R}^{n-k}$

foliation : (this is 1-dim)



$\mathbb{R}^k \times \mathbb{R}^{n-k}$

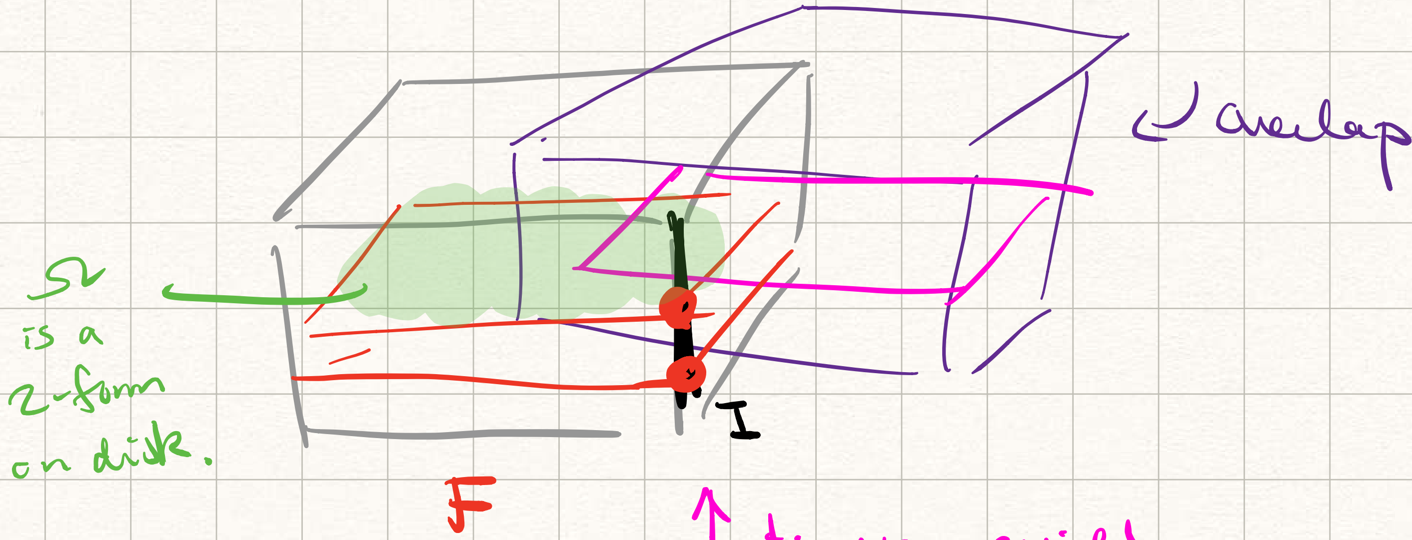
lamination :



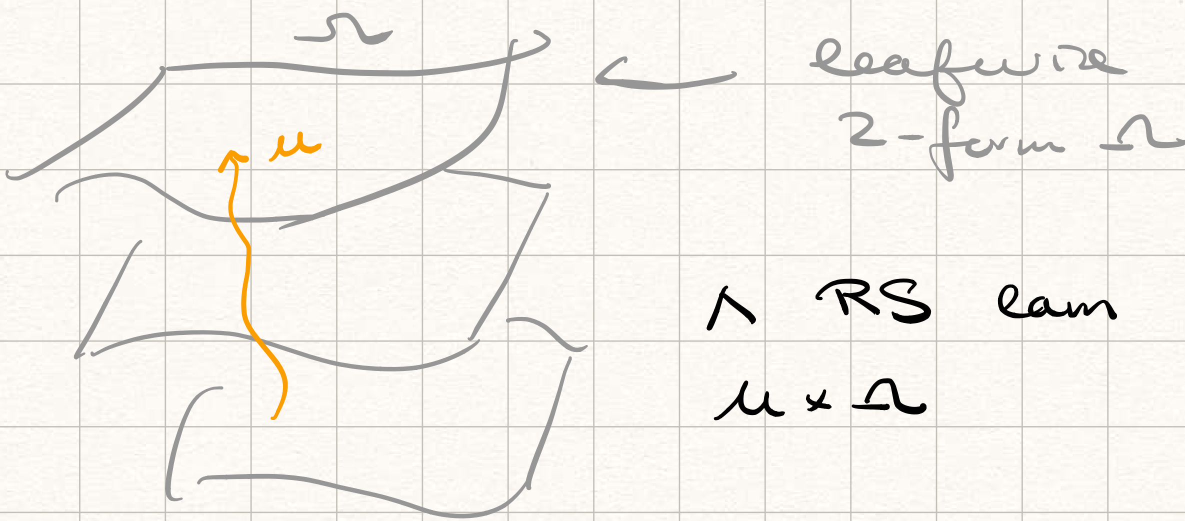
M^3, Λ RS lamination (called Λ).

Goal : Provide a criterion for (M, Λ) to admit a leafwise metric of ct curvature.

Def : (Invariant transverse)
A transverse measure μ for Λ is a non-neg Borel measure on the leaf space of Λ in small prod chart which is compatible on the overlap of distinct chart.



↑ pt; you will have overlaps of charts, & the measure better be compatible for overlaps.



Λ RS lam

$\mu \times \Omega$

$X(\mu)$ = total mass of $\mu \times \Omega$.

$$X(\mu) = [\mu \times \Omega](M)$$

as a measure on M.

Suppose Λ admits a leafwise invariant hyp. metric.

For each product chart $D \times I$,

$$\chi(\mu) = \left(\int_I \underbrace{\left(\int_D \Omega \right)}_{<0} d\mu \right) < 0$$

$$\Rightarrow \chi(\mu) < 0 \quad \forall \mu.$$

<u>Thm</u> (Candel)

(M, Λ) as before. Λ admits a leafwise hyp. metric
$\Longleftrightarrow \chi(\mu) < 0$ $\forall$ transverse meas. μ.

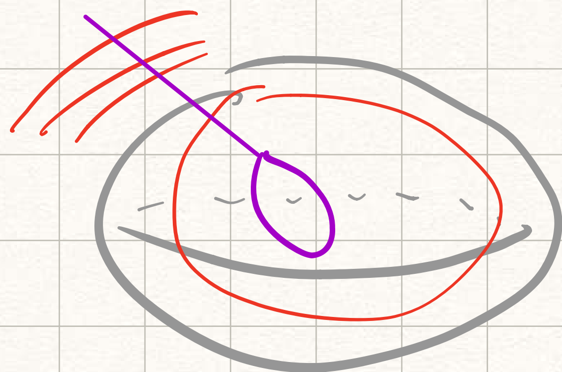
Consider $(\underset{\substack{\uparrow \\ \text{close}}}{M}, \underset{\substack{\uparrow \\ \text{taut foliation}}}{\mathcal{F}})$.

S^2 leaf $\Rightarrow$ they are S^2

Thm (Reeb Stability thm)

If $\mathcal{F}$ has S^2-leaf, then $M \cong S^1 \times S^2$ & $\mathcal{F}$ is the product foliation.

loop to make it taut

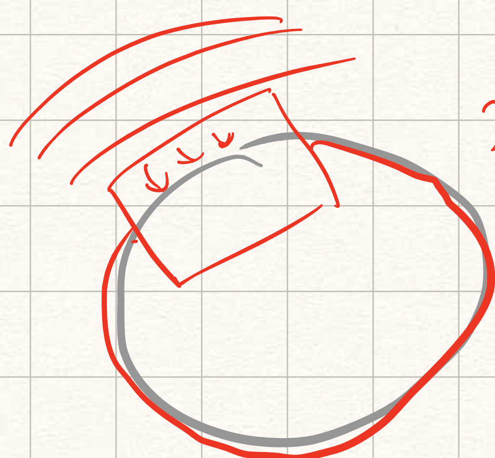


$\lambda \cong S^2$

$\pi_1(S^2) = 1$

$\Rightarrow$ λ has nbhd $\cong S^2 \times I$

$\Rightarrow$ The set of spherical leaves form an open subset of M.



$\lambda_i \cong S^2$

$\Rightarrow$ projection maps give a covering map $\lambda_i \to \lambda$ for large enough i's.

$\Rightarrow$ λ must be also S^2.

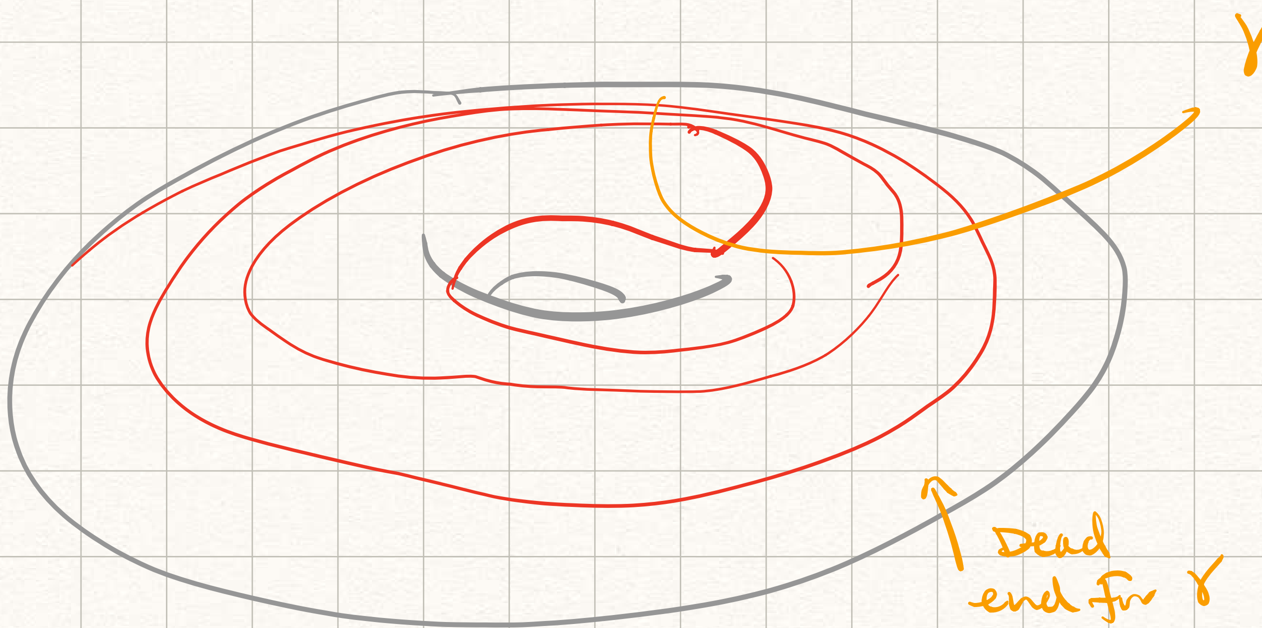
T^2 leaf $\Rightarrow$ they all hyp.

Consider $(M, \mathcal{F})$, where T^2 leaf,

$H = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$

foliated by planes $z = c$

$$H \setminus \{0,0,0\} \big/ (x,y,z) \sim (2x, 2y, 2z)$$



" ∞-many snake eats own tail ∞-many times."

$(M, \mathcal{F}) \Rightarrow$ all leaves of $\mathcal{F}$ are hyperbolic type

$\xRightarrow[\text{Candel}]{}$ $\mathcal{F}$ admits a leaf-wise hyperbolic metric.