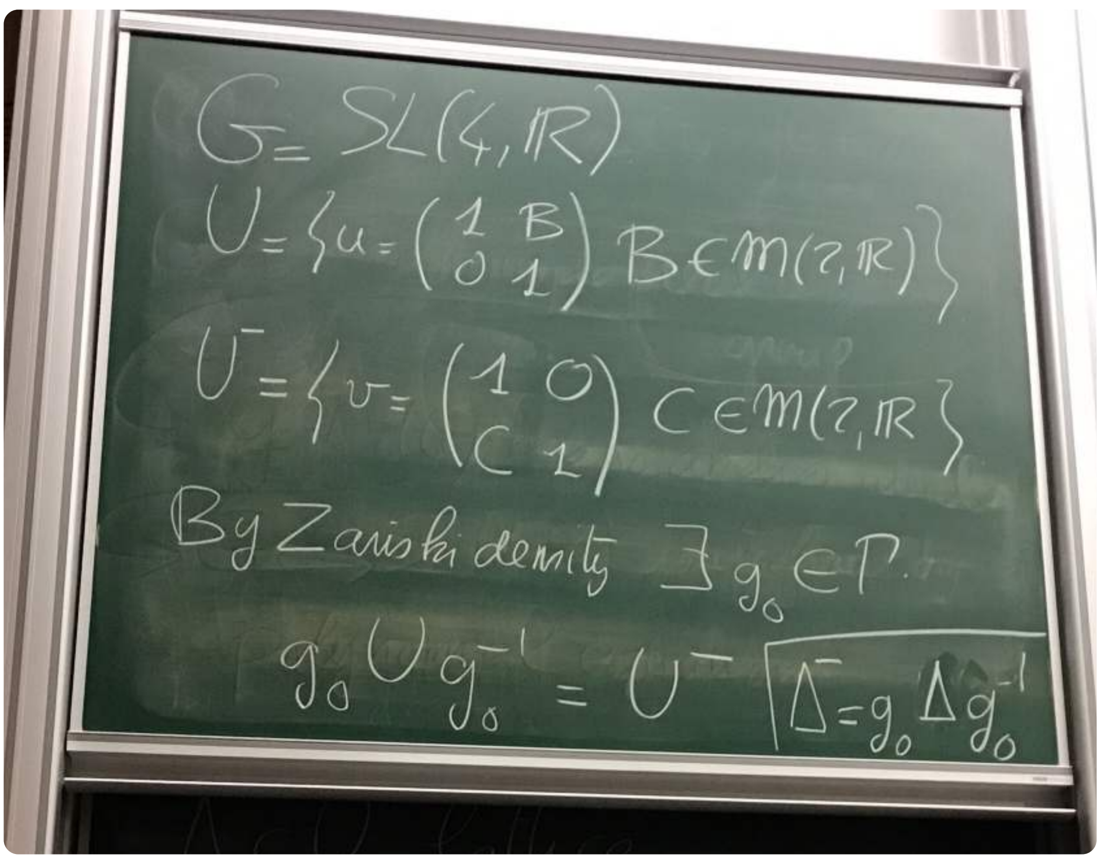
Goal: Prove the last thm about commensurability.

Rmk: the pt is that you don't assume lattice → assume discrete grp.

When Γ lattice, get "Margulis first arithmeticity thm".

"due to Hee Oh in many cases. including $G = SL(d, \mathbb{R})$



$\Delta \subset \mathcal{U}$ lattice

$\Delta^- \subset \mathcal{U}^-$ lattice

Thm: If $\Gamma = \langle \Delta, \Delta^- \rangle$ discrete, then Γ is commensurable to some $G_{\mathbb{Z}}$.

Plan to prove ~~this~~.

semi-simple $\to SL(4,\mathbb{R})$ ... Γ discrete grp containing lattice $\to$ is lattice.

Lecture 2 - Closeness of Orbit

Let $L = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| \det A \cdot \det D = 1 \right\}$ diagonal matrices in G.

Then $L \supset L_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| \det A, \det D = 1 \right\}$
$\simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})\}$

L_0 acts by conj on $\mathcal{U} \simeq M(2,\mathbb{R})$.
and on $\mathcal{U}^- \simeq M(2,\mathbb{R})$.

Let $X(\mathcal{U}) = \{$lattices of co-vol 1 in $\mathcal{U}\}$
$\simeq SL(4,\mathbb{R}) / SL(4,\mathbb{Z})$

$X(U^-) = \{$lattices of co-vol 1 in $U^-\}$.
$\simeq SL(4,\mathbb{R})/SL(4,\mathbb{Z})$

Key Prop (today).

a) The h_0-orbit $h_0\Delta$ is closed in $X(U)$

b) The h_0-orbit $h_0(\Delta,\Delta^-)$ is closed in $X(U)\times X(U^-)$

(tmrw) : closed $\Rightarrow$ finite vol
wrt-cmpt $\Rightarrow$ add elmt to Δ,Δ^- which are in L
(review).

② Strategy of pf of Key Prop:

$\underset{4}{g} = \underset{4}{U^-} \oplus \underset{7}{\ell} \oplus \underset{4}{U}$ in dim

$\pi : g \to U$ the projection

$\underline{Ad}(g) = 15$ dim. matrix. $\underline{\text{ex}}$: if $g = v\ell u$,
adjoint $\qquad M(g) = Ad\,\ell|_U$

For $g \in G$,

let $M(g) = \pi \, \mathrm{Ad}(g) \, \pi \in \mathrm{End}(\mathfrak{u})$

$$\mathrm{Ad}\, g = \begin{array}{c} \begin{array}{ccc} \mathfrak{u} & \mathfrak{l} & \mathfrak{u}^- \end{array} \\ \left(\begin{array}{c|c|c} \; & \; & \; \\ \hline \; & \; & \; \\ \hline \; & \; & M(g) \end{array}\right) \end{array}$$

$$\Omega = \mathfrak{u}^- L \mathfrak{u}$$

Lemma 1: $\{ M(g) X \mid g \in \Gamma \cap \Omega \mid X \in \Lambda \}$

$\Omega = U^- L U$ $\qquad$ $U \leftarrow \mathfrak{u} \simeq \mathbb{R}^4$

$\Lambda = \log(A)$ $\qquad$ $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \xrightarrow[\log]{} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$

is a closed discrete subset of $\mathfrak{u}$.

Lemma 2: Let $\Phi(g) = \det_{\mathfrak{u}}(M(g))$

Then $\Phi(\Gamma)$ is a closed discrete subset of $\mathbb{R}$.

Pf of Lemma 1 :

Let $x'_n = M(g_n)x_n$ & assume $x'_n \xrightarrow[n\to\infty]{} x'_\infty$. w/ $g_n = \underset{\underset{\mathcal{U}^-}{\cap}}{v_n} \underset{\underset{L}{\cap}}{\ell_n} \underset{\underset{\mathcal{U}}{\cap}}{u_n} \in \Gamma$, and $x_n \in \Lambda$.

I want $x'_n = x_\infty$ for $n >> 1$. (ie is closed)

Write $v_n = \delta_n^{-1} v'_n$, $\delta_n \in \Delta^-$ and $v'_n \to v'_\infty \in \mathcal{U}^-$.

Compute $\Gamma \ni \gamma_n$

$$\begin{aligned}\gamma_n &= \delta_n g_n e^{x_n} g_n^{-1} \delta_n \\ &= v'_n \ell_n e^{x_n} \ell_n^{-1} v_n^{-1} \\ &= v'_n e^{Ad\, \ell_n(x_n)} v_n^{-1} \\ &= v_n e^{x'_n} v_n^{-1} \\ &\xrightarrow[\infty]{} v'_\infty e^{x'_\infty} v'^{-1}_\infty\end{aligned}$$

then $v'_n e^{x'_n} v'^{-1}_n = v'_\infty e^{x'_\infty} v'^{-1}_\infty$

Apply π for $n >> 1$

then $x'_n = x'_\infty$ for $n >> 1$.

I think the discrete part is just b/c grps.

Pf of Lemma 2 :

Let $\Phi(g_n) \xrightarrow[n\to\infty]{} \Phi_\infty$.

I want $\Phi(g_n) = \Phi_\infty$ for $n >> \infty$.

w/ $g_n \in \Gamma$.

Look at lattices $M(g_n)\Lambda$.

$\Rightarrow$ By Lemma 1, $\bigcup_{n\geq 1} M(g_n)\Lambda$ is discrete set, & have bounded co-vol.

- By "Mahler crit", one can assume extracting $M(g_n)\Lambda \to \Lambda_\infty$ lattice in $\mathcal{U}$ (after extracting final subseq that converge).

Then $M(g_n)\Lambda \to \Lambda_\infty$ closed & discrete

not really right.

To be concrete :

For $g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$

$\Omega = \mathcal{U}^- L\, \mathcal{U} = \{ g \in G \mid \det A_g \neq 0 \}$

$\mathcal{U} \simeq M(2,\mathbb{R})$

$$X = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \longleftrightarrow B$$

So: $$M(g)X = \begin{pmatrix} 0 & A_g B D_g^{-1} \\ 0 & 0 \end{pmatrix}$$

$$\Phi(g) = (\det A_g)^4 .$$

Corol: Let $F(X) = \Phi(e^X g_0)$.

$\{F(X) \mid X \in \Lambda\}$ is a closed discrete subset of $\mathbb{R}$.

& $$F(X) = c_0 (\det B)^4$$

Pf of key prop (a)

Let $l_n \in h_0 \mid l_n \Lambda \to \Lambda_\infty$.

wts $\Lambda_\infty \in h_0 \Lambda$.

Write $l_n \Lambda = \varphi_n(\Lambda)$

where $\varphi_n \in SL(4, \mathbb{R})$ $\varphi_n \xrightarrow{n\to\infty} e$.

$\forall X \in \Lambda_\infty$, $X = \lim_{n\to\infty} \underbrace{\mathrm{Ad}\, l_n}_{\varphi_n}(X_n)$, $X_n \in \Lambda$

$$F(\varphi_n(X)) = F(\mathrm{Ad}\, \ell_n(X_n)), \quad X_n \in \Lambda$$
$$= F(X_n) \in F(\Lambda) \text{ which}$$
is discrete by lemma 2.

but $F(\varphi_n(X)) \longrightarrow F(X)$.

Then $F(\varphi_n(X)) = F(X)$ for $n \gg 1$.

For $n \gg 1$, $\forall X \in \Lambda_\infty$, ($\& \ \forall X \in \mathcal{U}$ by Z-density)

$$F(\underbrace{\varphi_n(X)}_{\text{we're saying this doesn't change det}}) = F(X)$$

$\Rightarrow \varphi_n \in h_0 \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

$\square$

Pf (b): Do the same for
$$G(X,Y) = \Phi(e^X e^Y).$$