Arithmeticity of Discrete grps

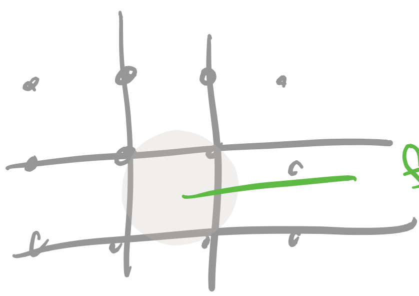
A historical survey

① Five examples:

i) $\Gamma = \mathbb{Z}^d$ is a lattice in $G = \mathbb{R}^d$,

ie Γ is a discrete lattice w/ co-volume $\mathrm{vol}(G/\Gamma) < \infty$ vol of fund domain finite

→ We call these lattices **co-compact.**

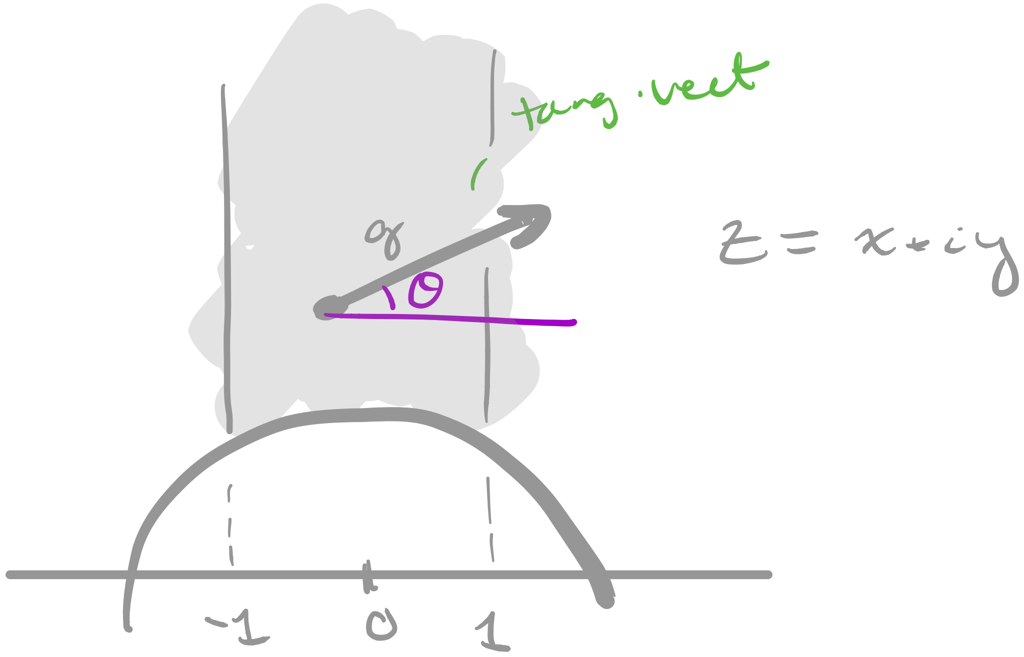


fundamental domain,

ii) $\Gamma = SL(d, \mathbb{Z})$ is a **non**-co-cmpt (Minkowski) lattice in $G = SL(d, \mathbb{R})$.

ex: $d = 2$

$G = T^1 \mathbb{H}^2$



$$dg = \frac{dx\,dy}{y^2}\, d\theta$$

iii) $\Gamma = SL(d, \mathbb{Z}[\sqrt{2}])$ is a non-cocompact lattice in $G = SL(d,\mathbb{R}) \times SL(d,\mathbb{R})$

key
$$g \mapsto (g, g^{\sigma})$$
where $(a, b\sqrt{2})^{\sigma} = a - b\sqrt{2}$

iv) (Siegel) $q = x_1^2 + \cdots + x_{d-1}^2 - x_d^2$

$\Gamma = SO(q, \mathbb{Z})$ is a non-cocompt lattice in $G = SO(q,\mathbb{R})$.

$$\Gamma \simeq \mathrm{Iso}(\mathbb{H}^{d-1})$$

v) $q = x_1^2 + \dots + x_{d-1}^2 - \sqrt{2}\, x_d^2$,

$\Gamma = SO(q, \mathbb{Z}[\sqrt{2}])$ is a co-compact lattice in

compact

$G = SO(q, \mathbb{R}) \times SO(q^\sigma, \mathbb{R})$

$g \longmapsto (g, g^\sigma)$

grp being compact $\Rightarrow$ removing it still preserves discrete & co-compact.

$G \approx \mathbb{H}^{d-1}$ pf: ~~this~~ is easy.

② General Results

G is a semi-simple real alg. grp.

Semi-simple: $\mathfrak{g} = \mathrm{Lie}(G)$,

ie has no abelian Ideals

ex: $G = SL(d, \mathbb{R})$,
$G = Sp(2d, \mathbb{R})$
$G = SO(q, \mathbb{R})$

Real alg. grp: $G \hookrightarrow SL(N, \mathbb{R})$ is defined by poly eq.

Thm: (Borel H...)
If G is defined over $\mathbb{Q}$, we can find the poly eq w/ coeffs in $\mathbb{Q}$,

then $G_{\mathbb{Z}} := G \cap SL(N, \mathbb{Z})$ w/ lattice in G.

Rmk: $\mathbb{Z}[\sqrt{2}] - \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} / a, b \in \mathbb{Z} \right\}$

Thm: (Borel density)
If $\Gamma \subset G$ lattice, then Γ is Zariski dense
i.e. all polynomial st

P st $P|_\Gamma = 0$ satisfy $P|_G = 0$.

Thm: (Kazdhan Margulis)

Let $\Gamma \subset G$ lattice, then

Γ co-compact $\Leftrightarrow$ Γ does not contain unipotent elmts

$(u-1)^N = 0$

Thm (Existence cusp)

Let $\Gamma \subset G$ a lattice.

If Γ is not compct, then $\exists\, U \subset G$ horospherical subgrp st:

$P \cap U$ co-compct in U.

$\exists\, g \in G$ st $\{u \in G \mid \lim_{n\to\infty} g^n u g^{-n} = e\}$

ex: $G = SL_4(\mathbb{R})$, $\left\{U = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \middle| B \in M(2,\mathbb{R})\right\}$

U is a horospherical subgrp.

Choose $g = \left(\begin{array}{c|c} \frac{1}{2} & 0 \\ \hline 0 & 2\mathbb{1} \end{array}\right)$

$$g^n \begin{pmatrix} A & B \\ C & D \end{pmatrix} g^{-n} = \begin{pmatrix} A & 4^{-n}B \\ 4^n C & D \end{pmatrix}$$

Thm (B. Miquel)

Let G be semi-simple w/ $rk_{\mathbb{R}}(G) \geq 2$.

- $U \subset G$ horospherical subgrp.
- $\Gamma \subset G$ is a discrete Zariski dense subgrp of G st

$\Delta = \Gamma \cap U$ is co-compact in U, and "irred"* in G.

| * $\Delta \cap G' = \{e\}$ $\forall\, G' \ntrianglelefteq G$.

Then Γ is commensurable* for some $G_{\mathbb{Z}}$ for some $\mathbb{Q}$-form of G.

($\star$): $\Gamma \cap G_{\mathbb{Z}}$ has finite index in both Γ and $G_{\mathbb{Z}}$.

$rk_{\mathbb{R}}(G) = \dim(\mathfrak{a})$,

$\mathfrak{a}$ = max abelian subalgebra of $\mathfrak{g}$ st $\forall\, X \in \mathfrak{a}$, ad X is diag over $\mathbb{R}$.,

$\mathrm{ad}X : \mathfrak{g} \to \mathfrak{g}$, $Y \mapsto [X, Y]$.

examples w/ $G = SL(4,\mathbb{R})$

- $\Gamma = SL(4,\mathbb{Z})$
- $\Gamma = SL(2, D_{\mathbb{Z}})$
- $\Delta = M(2,\mathbb{Z}) \simeq \mathbb{Z}^4$
- $D_{\mathbb{Z}}$ = quaternion ring st

  $D_{\mathbb{Z}} \otimes_{\mathbb{Z}} \simeq M(2,\mathbb{R})$

  $\Delta \simeq D_{\mathbb{Z}}$.

- "$\Gamma = SU(4, \mathbb{Z}[\sqrt{2}])$"

  $= \{ g \in SL(4, \mathbb{Z}[\sqrt{2}]) \mid gJ^{t}g^{\sigma} = J \}$

  w/ $J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Ans to what $SO(q, \mathbb{Z})$ is: