Orderability of 3-dim manifolds

Ref: Adam Clay, Dale Rolfsen.
Ordered grp & topology.

$G \neq \{1\}$. Say we have total ord on G. Then the pair $(G, <)$ is a left-order if $\forall g, h \in G$,

$$g < h \iff kg < kh \quad \forall k \in G.$$

If G admits a LO, it is LO'able.

<u>Convention</u>: $\{1\}$ is not LO.

If $(G, <)$ LO, then

$g \in G$ is $\begin{cases} \text{pos} & \text{if } g > 1 \\ \text{neg} & \text{if } g < 1. \end{cases}$

Easy consequences $(G,<)$ LO

① A non-trivial subgrp of G is LO

② $1<g \iff g^{-1}<1$

③ $g,h>1 \Rightarrow gh>1$ and
$g,h<1 \Rightarrow gh<1$.

④ If $g>1$, then $n\in\mathbb{Z}$, $g^{n+1}>g^n$.

$\Rightarrow$ G torsion-free.
(& infinite)

⑤ If $n>0$ then

$$\begin{cases} g>1 \iff g^n>1 \\ g<1 \iff g^n<1 \end{cases}$$

ex: $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$.

a) $\mathbb{R}$ usual ord.

b) $\mathbb{Z}$ has exactly two orders.

c) Every rational has a pow which is an integer
⇝ & the ord is det by positive elmt.
(just restr. on integers).
⇒ there are exactly two ords on $\mathbb{Q}$.

②

a) If $(G, <_G)$, $(H, <_H)$ ho then $G \times H$ is ho via

$$(g_1, h_1) < (g_2, h_2) \iff \begin{cases} g_1 <_G g_2 \text{ or} \\ g_1 = g_2, \\ h_1 < h_2 \end{cases}$$

Inductively, $\mathbb{Z}^n$ is ho $\forall n$
(& uncountably many b/c uncountably many basis).

b) More generally, if $\{(G_j, <_j)\}_{j\in J}$ a fam. of LO grps,

$\underset{i\in J}{\times} G_j$ admits LO as follows:

let $<$ be well-ord on J. Then define:

$$(g_j) < (g_j')$$

$$\Leftrightarrow g_{j_0} \underset{j}{<} g_{j_0}' \quad \text{where}$$

$$j_0 = \min_{<} (j \mid g_j \neq g_j')$$

ex: $\mathbb{R}$ VS over $\mathbb{Q}$ on uncountable dim

$\Rightarrow$ $\mathbb{R}$ has uncountably many LO's (by choosing pos basis elmt as a VS on $\mathbb{Q}$).

③ $\mathrm{Homeo}^+(\mathbb{R})$

$= \{ f : \mathbb{R} \to \mathbb{R} \mid x < y \Rightarrow f(x) < f(y) \}$

write $\mathbb{Q} = \{ r_1, r_2, \ldots, r_n, \ldots \}$

countable.

and embed.

$$\mathrm{Homeo}^+(\mathbb{R}) \longrightarrow \mathbb{R}^{\mathbb{N}}$$ put lex ord.

$$f \longmapsto (f(r_1), f(r_2), \ldots)$$

Exc: The total ord on $\mathrm{Homeo}^+(\mathbb{R})$ inherited from $\mathbb{R}^{\mathbb{N}}$ is an LO. $f(r_i) < g(r_i) \Rightarrow h(f(r_i)) < h(g(r_i))$

Exc: Suppose $(A, <_A), (B, <_B)$ are LO grps, and

$$1 \to A \xrightarrow{\psi} G \xrightarrow{\varphi} B \to 1$$

is exact.

Then, $g_1 < g_2$

$$\Leftrightarrow \begin{cases} 1 <_A \psi^{-1}(g_1^{-1} g_2) & \text{if } g_1^{-1} g_2 \in \psi(A) \\ 1 <_B \varphi(g_1^{-1} g_2) & \text{if } g_1^{-1} g_2 \notin \psi(A). \end{cases}$$

Countable lo grps

Thm (Conrad) A non-trivial countable grp is lo $\Leftrightarrow$ $G \leq \mathrm{Homeo}^+(\mathbb{R})$.

Pf: ($\Leftarrow$) Clear

($\Rightarrow$) Say $(G,<)$ lo
G countable

Say $G = \{1 = g_1, g_2, \ldots\}$. Construct embedding into $\mathbb{R}$ as follows:

$$f : G \to \mathbb{R},$$

Set $f(g_1) = 0$.

Suppose $f(g_1), \ldots, f(g_n)$ have been defined.

a) If $g_{n+1} < \min\{g_1, g_2, \ldots g_n\}$ then set

$$f(g_{n+1}) = \min\{f(g_1), \ldots, f(g_n)\} - 1$$

b) $g_{n+1} > \max\{g_1, \dots, g_n\}$ st

$$f(g_{n+1}) = \max\{f(g_1), \dots, f(g_n)\} + 1.$$

c) Otw choose $1 \leq j, k < n$ st
$g_j < g_{n+1} < g_k$ and

$\nexists\ 1 \leq i \leq n \ni g_j < g_i < g_k$

set $$f(g_{n+1}) = \frac{f(g_j) + f(g_k)}{2}.$$

Yields $f : G \to \mathbb{R}$ st $g < h$
$\Rightarrow f(g) < f(h)$.

<u>exc</u>: $f(G)$ is unbounded above & below.

Build action of G on $\mathbb{R}$ in three steps.

<u>Step 1</u>: G acts freely on $f(G)$
$$g \cdot f(g') := f(gg').$$

this is ord pres.

Step 2: Show action extends to one of G on $\overline{f(G)}$.

If $g \in G$ and $\{f(g_n)\}_{n \geq 1}$ a converging seq converging on $\mathbb{R}$.

Fact: $\{f(gg_n)\}_{n \geq 1}$ converges in $\mathbb{R}$.

$f(g_n) \to x \Rightarrow f(gg_n) = g f(x_n) \to gx$.

Define:

$$g \cdot \lim_n f(g_n) := \lim_n f(gg_n).$$

This defines a homeo of $\overline{f(G)}$.

Step 3 The component of $\mathbb{R} \setminus \overline{f(G)}$ is a disjt union of intervals (x,y) where $(x,y) \in \overline{f(G)}$

$\Rightarrow (1-t)x + ty \mapsto (1-t)gx + tgy$

Obtain $\varphi(g) : \mathbb{R} \xrightarrow{\cong} \mathbb{R}$ orient'n pres.

the correspondances
$$\varphi : G \longrightarrow \mathrm{Homeo}^+(\mathbb{R})$$
is an injective homomorphism.
b/c each $g \in G$ dif, act dif on ord.

φ is called the ==dynamic realization== of $(G, <)$.

exc: Use the fact that $f(G)$ is unbounded above & below in $\mathbb{R}$ to show that:

not yet but obvious.

$\nexists\, x \in \mathbb{R}$ st $\varphi(g)(x) = x \;\; \forall g \in G$.

Space of ho

Let $\{0,1\} = 2$ discrete top.
X space

$$2^X = \{f : X \to \{0,1\}\}$$

not nec. continuous

$$\equiv \prod_{x \in X} \{0,1\} \quad \text{w/ } f \mapsto (f(x))_{x \in X}.$$

Endow 2^X $\left(= \prod_{x \in X} \{0,1\}\right)$ w/ prod. top.

Thm:

① 2^X is compact (Tychonoff)
② 2^X is Hausdorff
③ Totally disconnected

If X is countably infinite, 2^X is the Cantor set.

Exc: $\forall x \in X$, show

$$\mathcal{U}_x = \{A \subseteq X \mid x \in A\}$$
$$\mathcal{U}_x^c = \{A \subseteq X \mid x \notin A\}$$

are open in 2^X.

Rmk: $2^X \equiv \mathcal{P}(X)$ the set of subsets of X.

$$f \longmapsto f^{-1}(1).$$

Positive Cones

If $P \subseteq G$ a grp, let

$$P(<) = \{g \in G \mid g > 1\}$$

"positive cone" of $<$.

$$P(<) \cdot P(<) \subseteq P(<).$$
$$G = P(<) \sqcup \{1\} \sqcup P(<)^{-1}.$$

Conversely, if $P \subset G$ satisfies

$$\begin{cases} PP \subset P \\ G = P \sqcup \{1\} \sqcup P^{-1} \end{cases}$$

then G admits LO def'd by $g < h \Leftrightarrow g^{-1}h \in P$. (exc).

$\Rightarrow$ get identification

$$LO(G) = \left\{ P \subset G \;\middle|\; \begin{array}{l} PP \subset P \text{ and} \\ G = P \sqcup P^{-1} \cup \{1\} \end{array} \right\}$$

$$\subseteq 2^G$$

$\rightsquigarrow$ $LO(G)$ inherits topology.

Prop. $LO(G)$ is closed in 2^G. Consequently, it is compact, Hausdorff and totally disc.

Pf: Observe that $LO(G) = S \cap C$ where

$$S = \{ A \subseteq G \mid AA \subseteq A \}$$
$$C = \{ A \subseteq G \mid G = A \sqcup A^{-1} \sqcup \{1\} \}$$

Clearly, $LO(G) = S \cap C$.

<u>wts</u>: S is closed.

$$2^G \setminus S = \left\{ A \subset G \;\middle|\; \begin{array}{c} \exists g,h \in A \text{ st} \\ gh \notin A \end{array} \right\}$$

$$= \bigcup_{g,h \in G} \left(U_g \cap U_h \cap U_{gh}^c \right)$$

which is open

$\Rightarrow S$ is closed. in 2^G.

Wts: C closed.

$$2^G \setminus C = \{A \subset G \mid G \neq A \sqcup A^{-1} \sqcup \{1\}\}$$
$$= \{A \subseteq G \mid \exists g \in G \text{ st } \{g, g^{-1}\} \subseteq A\}$$
$$= \bigcup \left\{ A \subseteq G \;\middle|\; \begin{array}{l} \exists g \in G - \{1\} \text{ st} \\ \{g, g^{-1}\} \cap A \neq \emptyset \end{array} \right\}$$

Exc: This is open.

$\Rightarrow C$ closed.

$\Rightarrow LO(G) = S \cap C$ closed.

The Bourns-Hale Thm

Consider a seq of balanced subsets B_k of a non-trivial grp G, ($B_k^{-1} = B_k$)

$$\{1\} \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_k \subseteq \cdots \subseteq \bigcup B_k = G$$

A proper k-partition (pkp) (wrt $\{B_k\}_{k \geq 1}$) is a $Q \subseteq B_k$

st $(GG) \cap B_k \subseteq G$

$B_k = G \sqcup G^{-1} \sqcup \{1\}$.

① pkp behaves like pos. cone on B_k.

Let $P_k = \{A \subseteq G \mid A \cap B_k \text{ a pkp of } G\}$

Then,

I don't get why...

- $P_1 \supseteq P_2 \supseteq P_3 \supseteq \dots \supseteq P_k \supseteq \dots$
- $P \in LO(G) \Rightarrow P \in P_k \ \forall k$

$\Rightarrow LO(G) \subseteq \bigcap_{k \geq 1} P_k$.

but basically yeah pos. cone need to behave like this.

Conversely, $P \in \bigcap_{k \geq 1} P_k \Rightarrow P \in LO(G)$.

↑ $\bigcup B_k$ exhaustive.

(exc).

- P_k is closed in 2^G.

<u>Idea</u>:

$P_k = S_k \cap C_k$ where

$S_k = \left\{ A \subseteq G \mid \begin{array}{l} \text{if } g,h \in A \cap B_k \text{ st} \\ gh \in B_k \text{ and } gh \in A \end{array} \right\}$

$C_k = \left\{ A \subseteq G \mid B_k = (A \cap B_k) \cup \{1\} \cup (A \cap B_k)^{-1} \right\}$

Show S_k, C_k closed in 2^G.

Lemma: $LO(G) \neq \emptyset$
$\Leftrightarrow P_k \neq \emptyset^k \quad \forall k \geq 1$.

Pf: Since 2^G compact,
$$LO(G) = \bigcap_{k \geq 1} P_k \neq \emptyset$$
by finite intersect. prop for compact spaces. *

* any finite set has non-empty intersect
(ex $\{P_k\}$ fin $\Rightarrow \bigcap P_k \neq \emptyset$.)

Prop: Say $G \neq \{1\}$. Then G is LO
$\Leftrightarrow$ each of its non-trivial finitely generated subgrps is LO.

Pf: $\Rightarrow$ ✓

$\Leftarrow$ Let F finite subset of G, $F \neq \{1\}$. Set

$$Q(F) = \left\{ Q \subseteq G \;\middle|\; \begin{array}{l} Q \cap \langle F \rangle \text{ is pos} \\ \text{cone for } F \end{array} \right\}$$

by assuming is ord for this fg subgrp $\langle F\rangle$.

By hypothesis, $Q(F) \neq \emptyset$.

exc: $Q(F)$ is closed in 2^G.

Given $F_1, \dots, F_n$ finite subsets $\neq \{1\}$.

$$\emptyset \neq Q(F_1 \cup F_2 \cup \dots \cup F_n) \subseteq \bigcap_{i=1}^{n} Q(F_i).$$

all sets Q st $Q \cap \langle F_1 .. F_n\rangle$ pos cone

each factor is cls restr.

just showed each

Then by finite int. prop, $\bigcap_{i=1}^{n} Q(F_i) \neq \emptyset$

$$\Rightarrow \bigcap_{\{1\} \neq F} Q(F) \neq \emptyset$$

But (exc) $\bigcap_{F} Q(F) = LO(G)$. $\square$

To get to Burns-Hale

1) Given $g_1, \dots, g_n \in G \setminus \{1\}$, let
$$S(g_1, \dots, g_n) = \langle g_1, \dots, g_n\rangle^+$$

→ Looks like a thm of Clay!

Thm: $G \neq \{1\}$ is LO $\iff$

$\forall \{g_1, \dots, g_n\} \subseteq G \setminus \{1\}$,
$\exists \{\varepsilon_1, \dots, \varepsilon_n\} \in \{\pm 1\}$ st
$1 \notin S(g_1^{\varepsilon_1}, g_2^{\varepsilon_2}, \dots, g_n^{\varepsilon_n})$.
$= \langle g_1^{\varepsilon_1}, g_2^{\varepsilon_2}, \dots, g_n^{\varepsilon_n} \rangle^+$

Thm (Burns-Hale)

Say $G \neq \{1\}$. Then G is LO
$\iff$ for every fg $H \leq G$, $H \neq \{1\}$,
$\exists$ epimorphism

$H \twoheadrightarrow L$ an LO grp.

ex: Free grps F are LO.

$1 \neq \underset{fg}{H} \leq F$.

Then H are fg free grp.
Then $H \twoheadrightarrow H/[H,H] \cong \mathbb{Z}^n$

some $n \geq$, $\mathbb{Z}^n$.

(we proved something almost like that
ie fg subgrp LO $\Rightarrow$ G LO).