G countable ho $\Leftrightarrow$ $G \leq \mathrm{Homeo}^+(\mathbb{R})$

<u>Thm</u>: $G \neq \{1\}$ is ho $\Leftrightarrow$ each $1 \neq \underset{\text{fg}}{H} \leq G$ is ho

(we proved this).

<u>Burns-Hale</u>: $G \neq \{1\}$ ho $\Leftrightarrow$ each $1 \neq H \leq G$ admits epimorphism $H \twoheadrightarrow L$ L LO.

(we won't prove this)

<u>Cor</u>:

- F_n is ho.
- $1 \to K \to A*B \to \overbrace{A \times B}^{\text{LO}} \to 1$

Kurosh $\Rightarrow K$ is free $\Rightarrow$ ho

$\Rightarrow$ $A*B$ ho (Vinograd).

3-Manifold grps

* irred.: any smooth sphere bounds a ball
non-ex: punctured.

$\pi_1(W)$, W compact, connected, orientable, irred.*

Thm (Boyer-Rolfsen-Wiest)
A 3-manifold grp $\pi_1(W)$ is LO $\Leftrightarrow$ $\exists$ epi

$$\pi_1(W) \twoheadrightarrow L, \quad L \text{ LO}$$

Note: Only requires epi of $\pi_1(W)$ and NOT every subgrp!
$\rightarrow$ Stronger than Burns-Hale.

Pf: ($\Rightarrow$) obvious
($\Leftarrow$) Let $\pi_1(W) \overset{\varphi}{\twoheadrightarrow} L$, LO.

Let $1 \neq \underset{fg}{H} \leq \pi_1(W)$.

Case 1: $[\pi_1(W) : H] < \infty$.
$\Rightarrow$ $\underset{\overset{\shortparallel}{L_0}}{\varphi(H)}$ has fi in L.

Then L LO $\Rightarrow h_0 \neq \{1\}$.

Then $\varphi|_H : H \twoheadrightarrow h_0$ LO

Case 2: $[\pi_1(W) : H] = \infty$.

Let $p: \tilde{W} \to W$ cover. w/

$\pi_1(\tilde{W}) = H$.

Then $\tilde{W}$ NOT compact as

$\deg(p) = \infty$.

Scott's Core thm:

$\Rightarrow$ $\tilde{W}$ contains a compact core

ie, $\exists$ compact, connected

$N \subseteq \mathrm{int}(\tilde{W})$ st

$\pi_1(N) \xrightarrow{\cong} \pi_1(\tilde{W})$.

Since $\tilde{W}$ not compact, $\partial N \neq \emptyset$.

(b/c take some ray $\to \infty$ in N, will

end on boundary as $\tilde{W}$ infinite).

wlog, choose N to minimize number of boundary components, $\# \text{comp } \partial N$. ← there can be more than 1 cmpt core.

Claim: $\nexists S^2 \subseteq \partial N$.

Pf: Assume not.

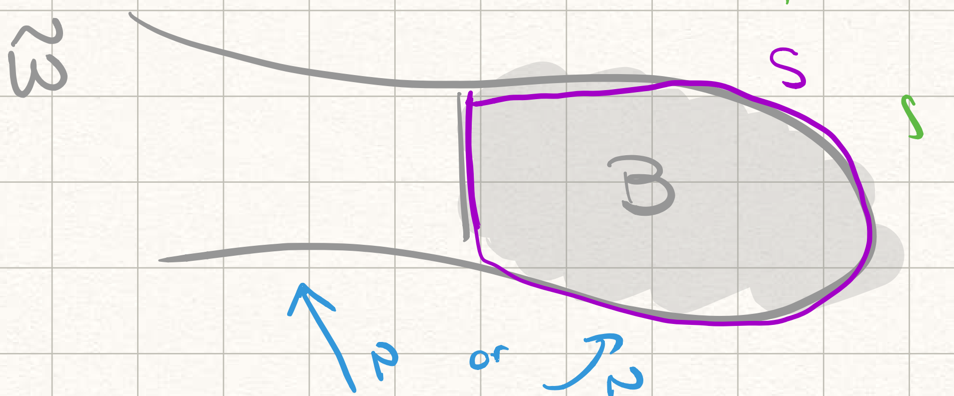
Let S boundary component of N that's a sphere

by some thm.

Otw note W $\underset{+\text{orient}}{\text{irred}}$ $\Rightarrow$ $\tilde{W}$ irred.

Then, $\exists$ 3-ball $B \subseteq \tilde{W}$ st

$\partial B = S$. every sphere bounds a ball in irred $\tilde{W}$



Either: $\begin{cases} N \subseteq B & \text{or} \\ N \cap B = S \end{cases}$ b/c N connected and $S \subseteq \partial N$.

Note: $N \subseteq B$ as otw

$$\begin{tikzcd}
\pi_1(N) \arrow[rr, "\cong"] \arrow[dr] & & \pi_1(\tilde{W}) = H \neq \{1\} \\
& \pi_1(B) \arrow[ur] &
\end{tikzcd}$$

b/c $N \subseteq B$,

$\pi_1 N$ would "factor through" $\pi_1(B)$ but can't w/c $\neq 1$.

So $N \cap B = S$

But then Seifert van Kampen
$\Rightarrow$

$$\begin{tikzcd}
\pi_1(N) \arrow[rr, hook, "\cong"] \arrow[dr, hook, "\cong"'] & & \pi_1(W) \\
& \pi_1(N \cup B) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Hence, $N \cup B$ has compact core w/ fewer ∂ comps than N.
b/c you just attached 1 bound-comp!

Contradiction!
by cmptness argument $\tilde{W}$.

But $\partial N \neq \emptyset$ so $\exists$ surface $F \subseteq \partial N$ of genus >0.

By std argument combining
- The long exact homology seq of $(N, \partial N)$
- Lefschetz duality
- Univ coefficient thm for cohomology

This implies that the first Betti number!

(*) $\Rightarrow \underset{\uparrow \text{ betti numbers.}}{b_1}(N) \geq \frac{1}{2} b_1(\partial N) \geq \frac{1}{2} b_1(F) \underset{\uparrow \text{ due to genus} > 0.}{>} 0$

So $b_1(N) > 0$. $b_1 = rk(H_1(N)) = k$, $\Rightarrow H_1(N) = \mathbb{Z}^k + tors.$

Hence, $\exists$ epimorphism

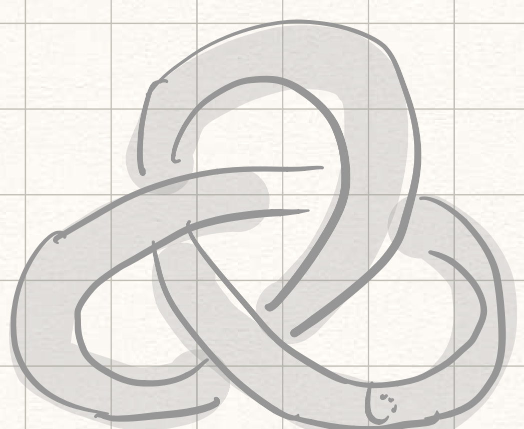
$$H = \pi_1(N) \twoheadrightarrow H_1(N) \twoheadrightarrow \mathbb{Z}$$

By Burns-Hale, $\overset{\text{Thm}}{\Rightarrow}$ $\pi_1(W)$ is lo $\square$

Cor: If W cmpt, oriented, irred, connected w/ $b_1(W) > 0$, then $\pi_1(W)$ is lo.

ex: ① K a knot in S^3.

Exterior of K: $\underbrace{S^3 \setminus \mathring{N}(K)}_{M_K}$



M_K compact, connected orientable $\underset{\text{thm}}{\text{irreducible}}$, and $b_1(M_K) = 1$.

So $\pi_1(M_K)$ is LO.

② Say W admits a coriented taut foliation $\mathcal{F}$ w/ leaf space $L \cong \mathbb{R}$.

$\pi_1(W)$ acts on $L = \mathbb{R}$ by elmts of $\mathrm{Homeo}^+(\mathbb{R})$.

Action is non-trivial (no global fixed pt).

$\Rightarrow \pi_1(W)$ is LO.

Corollary reduces question of whether $\pi_1(W)$ is LO to the case $b_1(W) = 0$.

Say $b_1(W) = 0$. by * otw $b_1(W) > 0$.
If $\partial W \neq \emptyset$, then $\partial W = \sqcup S^2$.
W irred $\Rightarrow W = B^3 \Rightarrow \pi_1(W) = 1$.
by S^2 bounds 3-ball.

Assume $\partial W = \emptyset$.

Then $b_1(W) = 0 \Rightarrow H_1(W)$ finite. ab. grp.

(by def)

$\Longleftrightarrow$ Poincaré duality $H_*(W, \mathbb{Q}) = H_*(S^3, \mathbb{Q})$.

ie, W is rat. hom. sphere ($\mathbb{Q}$HS)

Pt: The thm is very strong.

Seifert Fibre Spaces (SFS)

↖ Manifolds foliated by circle.

pt: irred not strong cond

Only reducible SFS are $S^1 \times S^2$ (and $P^3 \# P^3$.

↳ $\exists$ S^2 that doesn't bound 3-ball.

Let W $\mathbb{Q}$HS, SFS and w/ irred, compact, connected, orientable.

$$W^3 \xrightarrow{\pi} B^2, \quad \pi^{-1}(x) \cong S^1 \quad \forall x \in B.$$

foliated by circles

crush each circ to pt.

Away from finitely many circ, map triv. circ bundle.

$\exists$ finitely many circle fibers which are singular. Singularity measured by a positive ($a \geq 2$).

measuring how you fail to be triv. S^1 bundle.

★ I don't get what any of this means...

weight ≥ 2 when sing, at ∞ weight $= 1$, hit only the finitely many $a_i \geq 2$.

$$W \longrightarrow \mathcal{B} = \underbrace{B(a_1, a_2, \ldots, a_n)}_{\text{2-dim orbifold}}$$

$\exists$ exact sequence

$$1 \to \langle h \rangle \to \pi_1(W) \to \pi_1(\mathcal{B}) \to 1$$

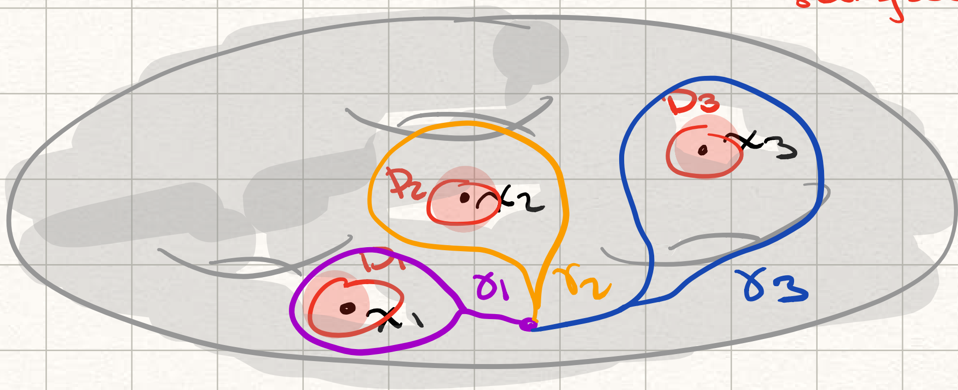
A? $\uparrow$ class of regular fiber.

ie $\pi_1(\mathcal{B}) = \pi_1(W)/\langle h \rangle$

$\mathcal{B}$, $n=3$

x_1, x_2, x_3 pt w/ exceptional fibers.

puncture surface.



$B_0 = \mathcal{B} \setminus \operatorname{int}(D_1 \cup D_2 \cup D_3)$,

$$\pi_1(\mathcal{B}) = \pi_1(B_0) \Big/ \langle\langle \gamma_1^{a_1}, \gamma_2^{a_2}, \gamma_3^{a_3} \rangle\rangle$$

usually $\leq PSL_2(\mathbb{R})$

eg. $\pi_1(S^2(a_1, a_2, a_3))$

$$\cong \left\langle \gamma_1, \ldots, \gamma_n \;\middle|\; \begin{array}{l} \gamma_1^{a_1} = \gamma_2^{a_2} = \cdots = \gamma_n^{a_n} = 1 \\ \gamma_1 \gamma_2 \cdots \gamma_n = 1 \end{array} \right\rangle$$

W a $\mathbb{Q}$HS

$$\Rightarrow 0 = b_1(W) \geq b_1(\mathcal{B}) \geq b_1(B) \geq 0$$

$\Rightarrow B = S^2$ or P^2.

$\Rightarrow \mathcal{B} = S^2(a_1, \dots, a_n)$ or $P^2(a_1, \dots, a_n)$.

Prop: W irred SFS QHS, then $\pi_1(W)$ is LO $\Leftrightarrow$

$\exists \rho : \pi_1(W) \to \mathrm{Homeo}^+(\mathbb{R})$ st $\rho(h) =$ translation by 1.

the class of reg. fiber again.

Pf: $\Leftarrow$ Then $\pi_1(W) \twoheadrightarrow \rho(\pi_1(W)) \leq \mathrm{Homeo}^+(\mathbb{R})$, so $\pi_1(W)$ is LO.

$\Rightarrow$ Say $\pi_1(W)$ is LO, then $\exists$ dynamical realization $\rho : \pi_1(W) \to \mathrm{Homeo}^+(\mathbb{R})$.

By construction, $\nexists x \in \mathbb{R}$ st $\rho(\gamma)(x) = x \quad \forall \gamma \in \pi_1(x)$

Claim: $\rho(h)$ has no fixed pt in $\mathbb{R}$.