W compact, connected, orient, irred

$\pi_1(W)$ 3-manif grp

$\pi_1(W)$ is LO $\Leftrightarrow$ $\exists$ epi $\pi_1(W) \twoheadrightarrow \mathbb{Z}$ LO

$\Leftrightarrow$ non-triv. homeo.

$\pi_1(W) \to \mathrm{Homeo}^+(\mathbb{R})$.

$b_1(W) > 0 \Rightarrow \pi_1(W)$ LO

$b_1(W) = 0 \Rightarrow \begin{cases} W \cong B^3 \text{ or} \\ W \text{ QHS.} \end{cases}$

SFS $W \to B = B(a_1, \dots, a_n)$

W QHS $\Rightarrow$ $B = S^2(a_1 \dots, a_n)$ or

$P^2(a_1 \dots a_n)$.

→ this sequence is for SFS's.

$1 \to \underline{\langle h \rangle} \to \pi_1(W) \to \pi_1(B) \to 0$

$B_0 = B^3 -$ pts, pre-ims into $\underline{W_0}$

where fibers are circles. locally zero bundle b/c removed the sings.

$h =$ loop going through fibers.

exactly once ⇒ this turns out to be well-def'd.

Lemma: W a SFS QHS.

Then $\pi_1(W)$ is LO

$\Leftrightarrow$ $\exists\, \rho : \pi_1(W) \to \mathrm{Homeo}^+(\mathbb{R})$

st $\rho(h) =$ trans. by 1.

Pf: ($\Leftarrow$) ✓

($\Rightarrow$)

Let $\varphi: \pi_1(W) \to \mathrm{Homeo}^+(\mathbb{R})$. The dynamic realization of some LO on $\pi_1(W)$.

It suffices to show that $\nexists$ fixed pt for $\rho(h) \curvearrowright \mathbb{R}$, ie

Challenge to show.

$\rho(h)(x) \neq x \quad \forall x \in \mathbb{R}$ b/c this $\Rightarrow$ $\rho(h)$ conj. to transl. by ± 1.

<u>I'll do the case $B = S^2(a_1, \ldots, a_n)$.</u>

Recall: $\pi_1(S^2(a_1, \ldots, a_n))$

$$= \langle \gamma_1, \ldots, \gamma_n \mid \gamma_i^{a_i} = 1, \gamma_1 \cdots \gamma_n = 1 \rangle$$

From $1 \to \langle h \rangle \to \pi_1(W) \to \pi_1(B) \to 1$,

we see

$$\pi_1(W) = \langle \gamma_1, \ldots, \gamma_n, h \mid \gamma_i^{a_i} = h^{b_i}, \gamma_1 \cdots \gamma_n = h^{b_0} \rangle.$$

Suppose fixed pt.

Suppose $\rho(h)(x_0) = x_0$. Then,

$$\rho(\gamma_i^{a_i})(x_0) = \rho(h^{b_0})(x_0) = x_0,$$

$$\rho(\gamma_i^{a_i}) = \rho(\gamma_i)^{a_i}$$

★ If $\rho(\gamma_i)(x_0) > x_0$
then $\rho(\gamma_i)^{a_i}(x_0) > \rho(\gamma_i)(x_0)$

Since $\rho(\gamma_i) \in \mathrm{Homeo}^+(\mathbb{R})$,

then $\overset{(★)}{\Longrightarrow}$ $\rho(\gamma_i)(x_0) = x_0$, $\forall i$

by prev. $\Rightarrow$ every elmt of $\pi_1(W)$ fixes x_0.

$\Rightarrow$ Contra.

$\hookrightarrow$ Dynamic realization doesn't have any global fixed pt.

Other case left as exc. $\square$

Prop: If $W \to P^2(a_1, \ldots, a_n)$, then $\pi_1(W)$ is NOT LO.

Pf: Have

$$\pi_1(P^2(a_1, \ldots, a_n)) = \langle \gamma_1 \ldots \gamma_n, \theta \mid \gamma_i^{a_i} = 1, \theta^2 \gamma_1 \ldots \gamma_n = 1 \rangle$$

This $\pi_1(W)$

$$= \langle \gamma_1 \ldots \gamma_n, \theta, h \mid [h, \gamma_i] = 1 \ \forall i, \ \gamma_i^{a_i} = h^{b_i}, \ \theta h^{-1} \theta = h^{-1}, \ \theta^2 \gamma_1 \ldots \gamma_n = h^{b_0} \rangle$$

If $\pi_1(W)$ is LO, $\exists \rho : \pi_1(W) \to \mathrm{Homeo}^+(\mathbb{R})$,

by prev prop $\rho(h) =$ trans. by 1.

Set $f^{-1} := \rho(\Theta) \in \mathrm{Homeo}^+(\mathbb{R})$,

then $\Theta h \Theta^{-1} = h^{-1}$ $\star$

$\Rightarrow f^{-1}(f(x)+1) = x-1$ orient pres.

$\Rightarrow f(x)+1 = f(x-1) < f(x)$.

$\Rightarrow$ Contra. assumpt: $\rho(h)(x) = x+1$.

$\star \Rightarrow \rho(\Theta)\rho(h)\rho(\Theta)^{-1} = \rho(h)^{-1}$ $\square$

$\Rightarrow f^{-1}\rho(h)[f(x)] = \rho(h)^{-1}(x)$

$= f^{-1}(f(x)+1) = x-1$.

Rmk:

① $W \to S^2(a_1, \dots, a_n)$

Work of Jankins-Neumann and of N— determine exactly when $\exists\, \rho : \pi_1(W) \to \mathrm{Homeo}^+(\mathbb{R})$ st $\rho(h) = 1$ in terms of the integers $a_1, a_2, \dots, a_n, b_1, \dots b_n$.

② E-H-N showed that W admits a co-oriented hor. foliation $\Leftrightarrow \exists$ such a ρ

foliations are everywhere transverse.
circ are loops

Thus, for SFS $\mathbb{Q}$HS, TFAE

1. $\pi_1(W)$ is LO
2. W admits a co-oriented taut foliation.
3. W is not a Heegaard-Floer L-space. (☆)

(☆) <u>Lisca-Stipsicz</u>:
If $W \to S^2(a_1 \ldots a_n)$, then W is an L-space.
$\Leftrightarrow$ $\nexists$ horizontal foliation.

<u>B-Gordon-Watson</u>: If $W \to P^2(a_1 \ldots a_n)$, then W is an L-space.

Left orderability and $\mathrm{Homeo}^+(S^1)$

Define $\widetilde{\mathrm{Homeo}^+(S^1)} \leq \mathrm{Homeo}^+(\mathbb{R})$
just by lifting circle.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd} \qquad \begin{tikzcd} \tilde{f} \in \widetilde{\mathrm{Homeo}^+(S^1)} \arrow[d] \\ f \in \mathrm{Homeo}^+(S^1) \end{tikzcd}$$

$\exists$ exact sequence

basically mod by $\mathbb{Z}$ to get this.

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}^+(S^1)} \to \mathrm{Homeo}^+(S^1) \to 1.$$

central gen'd by trans. by 1.

Say

can you get a rep. lift?

$$\begin{tikzcd} & \widetilde{\mathrm{Homeo}^+(S^1)} \arrow[d] \\ \pi_1(W) \arrow[r, "\rho"] \arrow[ur, dashed, "\exists \tilde{\rho}?"] & \mathrm{Homeo}^+(S^1) \end{tikzcd}$$

$e(\rho) \in H^2(W)$ Euler class.

↳ measures how hard is it to get the right thing.

Fact: $\exists \tilde{\rho} \Leftrightarrow e(\rho) = 0$.

so like when herdless $=0$.

Problem: Construct

$$\rho : \pi_1(W) \to \mathrm{Homeo}^+(S^1)$$

w/ $e(\rho) = 0$.

If one exists, $\pi_1(W)$ is LO.

Say W is an $\mathbb{Z}HS^3$

ie, $H_*(W) \cong H_*(S^3)$

$$\left(\iff H_1(W) = 0 \right).$$

In this case,

$$H^2(W) \cong H_1(W) = 0.$$

↑ Poincaré duality

Then, $e(\rho) = 0$

$\forall \rho : \pi_1(W) \to \mathrm{Homeo}^+(S^1)$

ex: If W admits a co-orient,

→ taut foliation, or

→ a q-geod flow, or

→ a pseudo-A flow.

then $\exists$ universal circ. rep:

$\rho : \pi_1(W) \to \mathrm{Homeo}^+(S^1)$

$\Rightarrow \pi_1(W)$ is lo.

<u>ex</u>: W a SFS $\mathbb{Z}$HS,

$H_1(W) = 0$

$\Rightarrow B = S^2(a_1, \ldots, a_n)$

and $\gcd(a_i, a_j) = 1 \ \forall i \neq j$.

Conversely, given $a_1 \ldots a_n \geq 2$, st

$\gcd(a_i, a_j) = 1 \quad i \neq j$,

$\exists!$. SFS $\mathbb{Z}$HS W st

$W \to S^2(a_1, \ldots, a_n)$.

In this case,

$$\pi_1(S^2(a_1, \ldots, a_n)) = \langle \gamma_1 \ldots \gamma_n \mid \gamma_i^{a_i} = 1, \gamma_1 \gamma_2 \ldots \gamma_n = 1 \rangle$$

$$\cong \begin{cases} \{1\} & \text{if } n \leq 2 \\ \text{icosahedral grp} & \text{if } (a_1 \ldots a_n) = (2,3,5) \\ \leq PSL_2(\mathbb{R}) & \text{otw.} \end{cases}$$

Prop: W a SFS $\mathbb{Z}$HS other than S^3 or $\Sigma(2,3,5)$.

which is Poincaré homology sphere.

Then, $\pi_1(W)$ is LO.

Pf: It is known that

$$\pi_1(S^2(a_1 \ldots a_n)) = \begin{cases} \{1\} \Rightarrow W = S^3 \\ \text{isocosahedral grp} \Rightarrow W = \Sigma(2,3,5) \end{cases}$$

thus can assume

$$\pi_1(S^2(a_1 \ldots a_n)) \leq PSL_2(\mathbb{R}),$$

but: $PSL_2(\mathbb{R}) \leq \mathrm{Homeo}^+(S^1)$, so we have

$$\pi_1(W) \twoheadrightarrow \pi_1(S^2(a_1 \ldots a_n)) \to \mathrm{Homeo}^+(S^1)$$

$\Rightarrow \pi_1(W)$ is LO

$\square$.

Conjecture If W irred $\mathbb{Z}HS$ other than S^3, $\Sigma(2,3,5)$. Then, $\pi_1(W)$ is ho.

Euler Class of universal circ. repn.

Prop: W irred $\mathbb{Q}HS$

① If $\mathcal{F}$ a co-orient. taut foliation on W and ρ its associated univ. circ. rep.

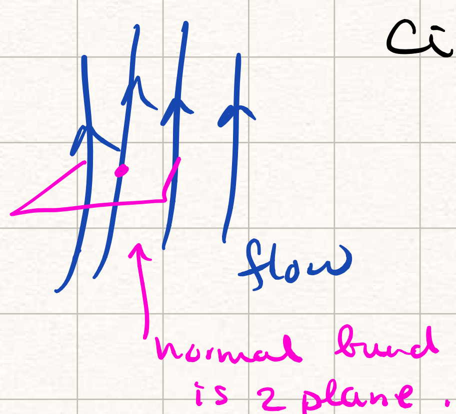
Then, $e(\rho) = e(T\mathcal{F}) \in H^2(W)$

tangent bundle of $\mathcal{F}$.

② If W admits a q-geod flow, a pseudo-A flow Φ and the ρ the assoc'd univ. circ. map, then

$$e(\rho) = e(\nu_\Phi)$$

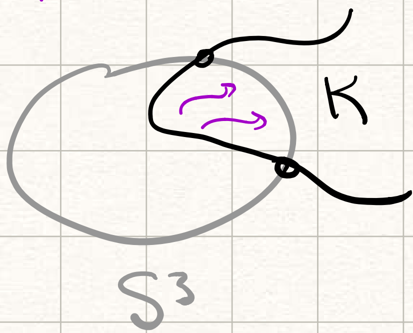
normal bundle to flow



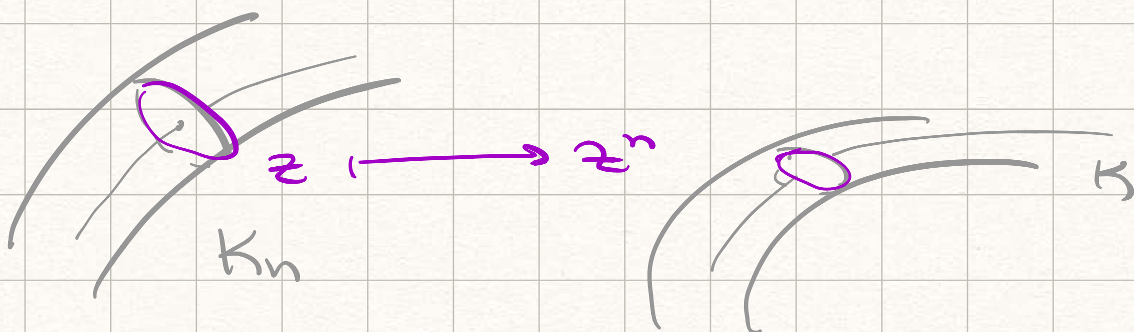
ex: If K is a fibred hyp, strongly q-pos knot. Then $\Sigma_n(K)$ has LO π_1.

n-fold branched cyclic cover of S^3 branched over K

$\forall n \geq 2$



$$\Sigma_n(K) \longrightarrow S^3$$



Suppose K trefoil.

$\pi_1(K(p/q))$ p/q surgery on K.

The L-space conjecture

Let W be closed, conn, orient, irred. TFAE:

① LO: $\pi_1(W)$ is LO (grp. theoretic).

② CTF: W admits a co-orient taut foliation (topological)

③ NLS: W is not a HF L-space (analytic)

CONJ

Evidence:

① True if $b_1 > 0$.
② True if SFS
③ True graph manif.
④ Nathan Dunfield looked at over 600K hyp. manifold. from census & no contra. yet.

exs:

① Most surgeries on alternating knots are not L-spaces and CTF. Show they are LO. NLS

② If K is a positive L-space knot, then $K(p/q)$ NLS $\Leftrightarrow \frac{p}{q} < \underbrace{2g(K)}_{\text{genus of } K} - 1$.

Show $K(p/q)$ is LO (resp. CTF) $\Leftrightarrow p/q < 2g(K) - 1$.

③ If K fib'd hyp surf q-pos, then $\Sigma_n(K)$ LO $\forall n \geq 2$.
Show $\Sigma_n(K)$ CTF, NLS for $n \geq 2$.