Thompson Groups

F as a grp of fractions

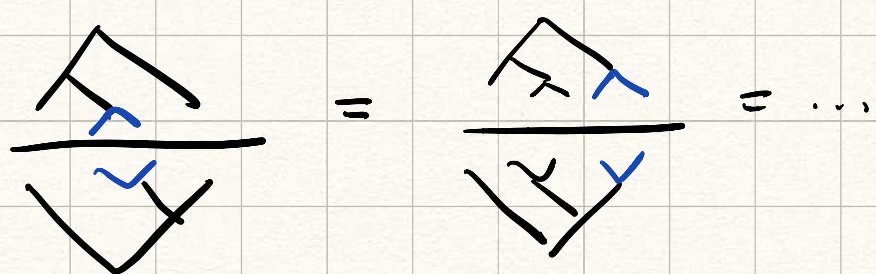
$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9}$$

$$\frac{2}{1} = \frac{4}{2} = \frac{6}{3}$$

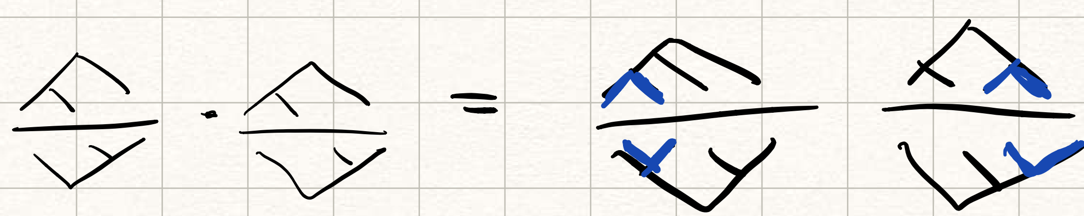
if we know eq. fract:

$$\frac{2}{3} \cdot \frac{2}{1} = \frac{4}{\cancel{6}} \cdot \frac{\cancel{6}}{3} = \frac{4}{3}$$

analogously:

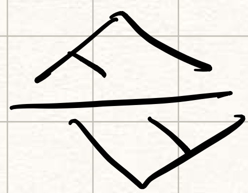


"table lookup method of mult. pair of trees":

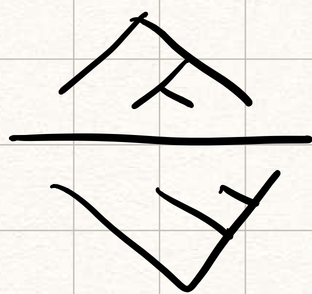


Def: F is the grp consisting of all tree fractions.

Exc: F is f.g. by two elmts,

$x_0 =$ 

&

$x_1 =$ 

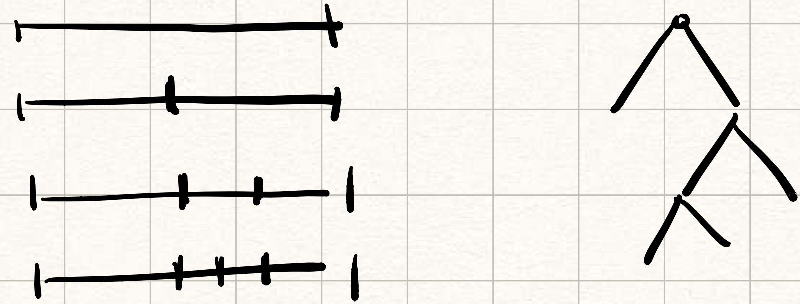
Hint: interpret these x_0, x_1 as rewriting rules for parenthesized expressions.

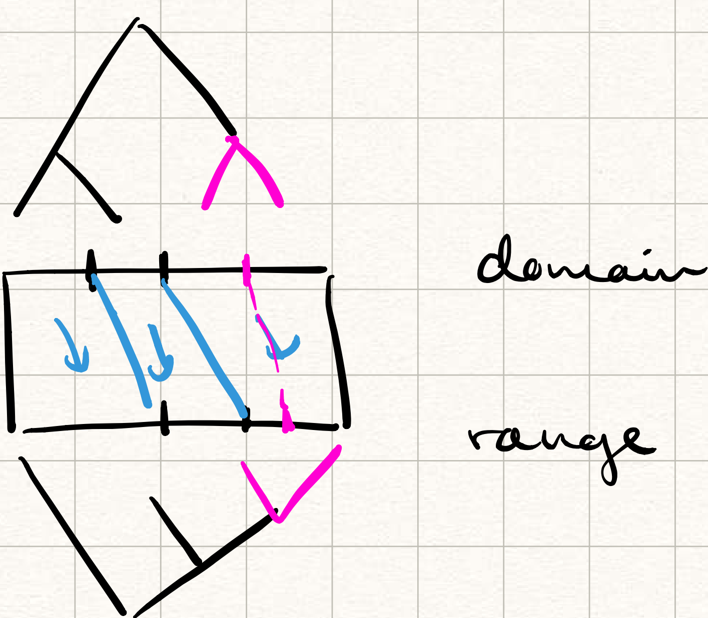
$$(AB)C \underset{x_0^{-1}}{\overset{x_0}{\rightleftarrows}} A(BC)$$

$$A((BC)D) \underset{x_1^{-1}}{\overset{x_1}{\rightleftarrows}} A(B(CD))$$

II - Representations as homeomorphism grps

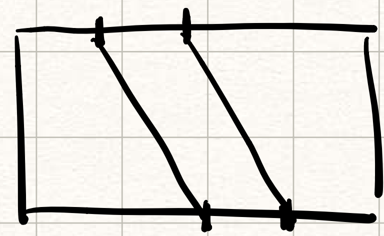
Obs: A binary rooted tree defines a binary subdivision of an interval or the Cantor set.

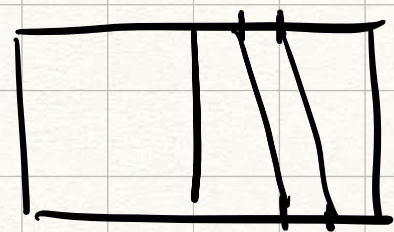




Obs: $\left.\begin{array}{l} F \to \mathrm{Homeo}(I) \\ F \to \mathrm{Homeo}(\mathcal{C}) \end{array}\right\}$ are injective.

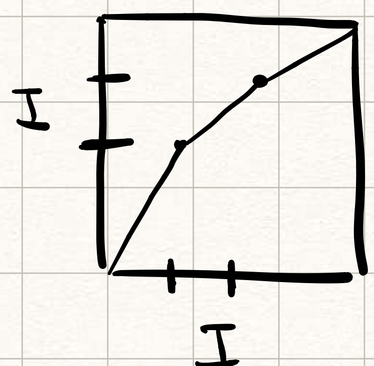
$\mathcal{C}$ cantor set

Now, $x_0 =$ 

$x_1 =$ 

exc: Show that F is torsion-free, infinite and not abelian.

Another way to map is:

x_0: 

Obs: Each homeo $f: I \to I$ coming from f has breaking pts in the dyadic rationals and slopes in $2^{\mathbb{Z}}$

Obs (Chain rule):

Here is a homomorphism

abeliani-zation

$$\mathrm{ab}: F \to \mathbb{Z} \times \mathbb{Z}$$

$$f \mapsto \Big(\underbrace{\log_2 f'(0)}_{\mathrm{ab}_\ell(f)},\ \underbrace{\log_2 f'(1)}_{\mathrm{ab}_r(f)}\Big)$$

left   right

$$x_0 \overset{\mathrm{ab}}{\longmapsto} (1, -1), \quad x_1 \overset{\mathrm{ab}}{\longmapsto} (0, -1)$$

1. If $f \in F$ is given as a word in x_0 and x_1, its exponent sum is equal to $-\mathrm{ab}_r(f)$.

2. $\mathrm{ab}: F \to \mathbb{Z} \times \mathbb{Z}$ is the abelianization and

$$[F, F] = \ker \mathrm{ab} = \{ f \in F \mid \underbrace{f'(0) = 1}_{\text{will go to 0 in log.}} = f'(1) \}.$$

$\square$

Def: Let $J \subseteq I$ be an interval w/ endpts dyadic rationals.

Define:

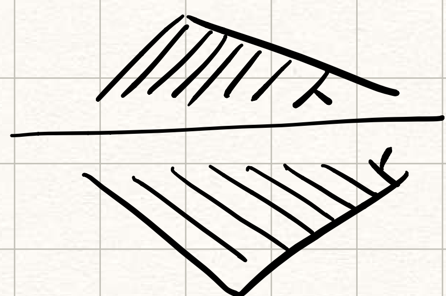
$$F(J) := \{f : I \to I \in F \mid f_{I \setminus J} = \mathrm{id}\}.$$

Fact: $F(J)$ is isom. to F.

Cor: There are many copies of F in $[F, F]$.

III - The infinite generating set

$x_n :=$



$n+3$ leaves in each tree

exc: Show that $x_i^{-1} x_m x_i = x_{m+1}$ for $i<m$.

Fact: $F = \langle x_0, x_1, x_2, \ldots \mid x_m x_i = x_i x_{m+1} \text{ for } i<m \rangle$

Prop: Let $f \in F$ be rep'd by a word in x_0, x_1 w/ exponent sum $s = -ab_r(f) \neq 0$.

Then the normal closure of f contains $[F, F]$.

Pf: Wlog $s > 0$. For very large m:

x review

$$f^{-1} x_m f = x_{m+s}.$$

bunch of x_0, x_1 conj.

Let N be the normal closure of f.

thus: $x_m \equiv x_{m+s} \mod N$

$x_1 \equiv x_{1+s} \mod N$

$x_1 \equiv x_{1+s+1} \mod N$

where $a \equiv b \mod N$
$\iff aN = bN$.

then:

$x_1 + s \equiv x_2 + s$

$x_1 \equiv x_2 = x_0^{-1} x_1 x_0$

$\Rightarrow x_0$ and x_1 commute mod N.

ie, F/N is abelian.

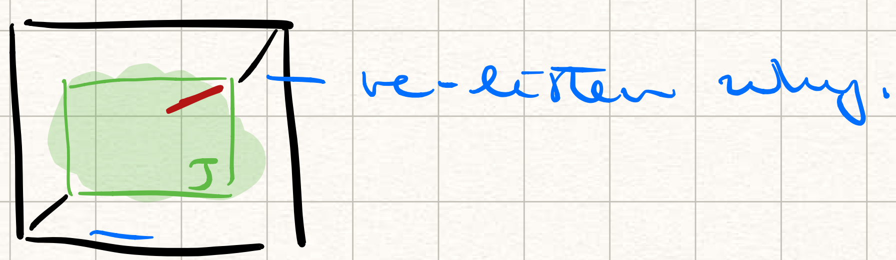
IV - Thm $[F,F]$ is simple.

Pf: Let $f \in [F,F]$ non-trivial and let N be its normal closure.

wts: $N = [F,F]$.

Let $J := \mathrm{supp}(f) := \{x \in I \mid f(x) \neq x\}$

Let's pretend J interval.

f: 

$f \in F(J)$. Within $F(J)$,

$ab_r^J(f) \neq 0$.

? thus: $[F(J), F(J)] \subseteq N$.

The following exc completes pf:

exc: Show that any elmt of $[F,F]$ is conjugate in $[F,F]$ to an elmt of $[F(J), F(J)]$.

Rmk: F has trivial center

Thus, every proper quotient of F is abelian. (why?)