# Affine Spaces and its automorphisms

- $k$ - field
- $\mathbb{A}^m_k$ = affine space over $k$
- $(x_1, x_2, \ldots, x_m)$ = std affine coord.
- $\mathbb{A}^m_k(S)$ = pts w/ coords in $S$.

ex: $\mathbb{A}^m(\mathbb{Z}) \simeq \mathbb{Z}^m$

$\mathbb{A}^m(\mathbb{C}) \simeq \mathbb{C}^m$

$\mathrm{End}(\mathbb{A}^m_k)$ polynomial transfo

$f: \mathbb{A}^m \to \mathbb{A}^m$.

$f(x_1, \ldots, x_m) = (f_1, \ldots, f_m)$

$f_i \in \underbrace{k[x_1, \ldots, x_m]}_{\text{poly fn.}}$

<u>Note</u>: $f \circ g = (f_1(g_1, \ldots, g_m), \ldots)$

$\mathrm{Aut}(\mathbb{A}^m_k)$ = invertible elmts in $\mathrm{End}(\mathbb{A}^m_k)$

= grp of automorphisms of $\mathbb{A}^m_k$ defined over $k$

Ex: $m \geq 1 \Rightarrow \mathrm{Aff}_m(k)$ = affine trans.

$$= GL_m(k) \ltimes \underbrace{k^m}_{\mathbb{A}^m(k)}$$

$$(x_1, \dots, x_m) \mapsto L(x_1, \dots, x_m) + t$$
$$\subset \mathrm{Aut}(\mathbb{A}^m_k).$$

exc 1: $\mathrm{Aut}(\mathbb{A}^1_k) = \mathrm{Aff}_1(k)$

$f \in \mathrm{Aut}(\mathbb{A}^1_k)$

$f(x_1) \in k[x_1]$, $f = ax_1 + b$

$f^{-1}(x_1) \in k[x_1]$ $\in \mathrm{Aff}_1(k)$

$f \circ f^{-1}(x_1) = x_1$

$\deg(f) \times \deg(f^{-1}) = 1.$

## 2. Dimension 2

ex: $h\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a x_1 + p(x_2) \\ b x_2 + c \end{pmatrix}$

$p \in k[x_2]$

$h^{-1} = \begin{pmatrix} * x_1 - * p(x_2) \\ * x_2 + * \end{pmatrix}$

$a, b, c \in k, \quad ab \neq 0$

This is called the

elementary subgrp

$$= \left\{ h = \begin{pmatrix} a x_1 + p(x_2) \\ b x_2 + c \end{pmatrix} \right\}$$

## Deg of endo:

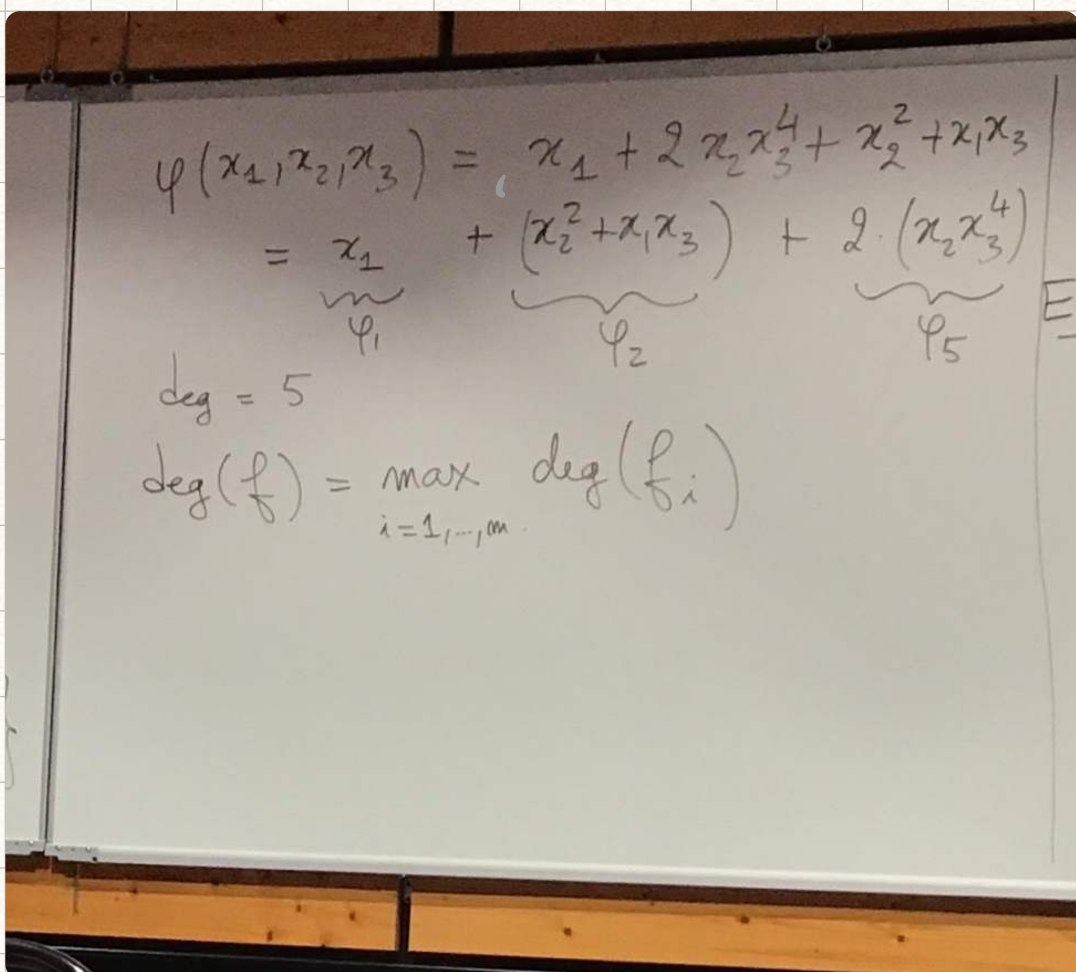
$$f(x_1,\dots,x_m) = (f_1,\dots,f_m)$$

$$\varphi \in k[x_1,\dots,x_m]$$

$$\varphi(\vec{x}) = \sum_{j=0}^{\infty} \varphi_j(\vec{x}) \quad \text{where}$$

$\varphi_j$ = hom gen pol fn deg $j$.

$$\deg \varphi = \max\{j \,;\, \varphi_j \neq 0\}.$$



## Geom. interpretation

Take a generic affine hyperplane:
$H \subset \mathbb{A}^m_k$.

Take generic line $L \subset \mathbb{A}^m_k$.
Compute:

$$\underbrace{f^{-1}(H) \cap L}_{\substack{\text{count \# pts} \\ \text{over } \bar{k} = \text{alg. closure}}} = \deg f$$

**Exc:** If $h_1$ and $h_2 \in E$,
then

$$\deg(h_1 \circ h_2) \leq \max(\deg h_1, \deg h_2).$$

**Thm:** [Jung, van der Kulk]

The grp of $\mathrm{Aut}(\mathbb{A}^2_k)$ is the free product of

$$A = \mathrm{Aff}_2(k)$$

and $E =$ elmt transfo.

amalgamated along their intersection.

$$S = A \cap E.$$

$$S = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} s \\ t \end{pmatrix}$$

w/ $L = \begin{pmatrix} * & \alpha \\ 0 & * \end{pmatrix} \in GL_2(k)$.

$$\mathrm{Aut}(\mathbb{A}^2_k) = A *_S E.$$

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$\forall h \in \mathrm{Aut}(\mathbb{A}^2_k) \setminus S$

$\exists g_1, \dots, g_n \in (A \cup E) \setminus S$ st

$$h = g_n \circ \dots \circ g_1$$

and two consecutive $g_i, g_{i+1}$ are in distinct subgrps $A, E$.

→ call this a reduced word (decomposition).

As soon as you write such a product, the result is not id.

**Prop**: The degree of a red. word is

$$\boxed{\deg(h) = \prod_{i=1}^{e} \deg(g_i)}$$

**Pf**: $$h = a_n \cdots \overbrace{e_3 \circ a_2 \circ e_2}^{E \setminus S} \circ a_1 \circ e_1$$

$A \setminus S$

$$\deg(h) \stackrel{?}{=} \prod \deg(e_i) \,?$$

Rmk: Every elmt of $A \setminus S$ can be written

$$s_1 \circ t \circ s_2$$

where $s_i \in S$, $t\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$.

$\Rightarrow$ $S \cdot t \cdot S = A \setminus S$.

$$h = \ldots e_3' \circ t \circ e_2' \circ t \circ e_1'$$

and $\deg(e_i') = \deg(e_i)$ $\forall i$.

$$e_1' \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 x_1 + p_1(x_2) \\ b_1 x_2 + c_1 \end{pmatrix}$$

$$\deg(e_1') = \deg(p_1)$$

$$t \circ e_1' = \begin{pmatrix} b_1 x_2 + c_1 \\ a_1 x_1 + p_1(x_2) \end{pmatrix} = \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix}$$

Now, $e_2'(t \circ e_1') = \begin{pmatrix} * + \overset{\text{leading.}}{p_2(Q_1)} \\ * + * \end{pmatrix}$

$$p_2(Q_1) = p_2(\underbrace{a_1x_1 + p_1(x_2)}_{Q_1})$$

$$\deg(p_2 \circ Q_1) = \deg(p_2) \times \deg(Q_1)$$
$$= \deg(e_2') \times \deg(e_1')$$

$$t \circ e_2' \circ t \circ e_1' = \begin{pmatrix} P_2 \\ Q_2 \end{pmatrix}$$

$\rightsquigarrow$ recursion $\square$

**Thm** [Lamy]

$k = \mathbb{C}$. The grp $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ satisfies the Tits-alternative:

If $\Gamma < \mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ is fg, then either:

i) $\Gamma$ contains a finite index solvable subgrp

OR

( ii) $\Gamma$ contains a non-abelian free grp )

$\underline{Q}$: What about Tits-alt for $\mathrm{Aut}(\mathbb{A}^m_{\mathbb{R}})$? $m \geq 3$.

We have ? for $\mathrm{Out}(F_n)$
& $\mathrm{MCG}(\Sigma_g)$.

I think is what he said.

## $\underline{\text{3 - Degree growth}}$

$\underline{\text{3.A Prop (exc)}}$: If $h \in \mathrm{Aut}(\mathbb{A}^2_{\mathbb{R}})$, then either

- $n \mapsto \deg(h^n)$ is bounded

or

- $n \mapsto \deg(h^n)$ grows like $\lambda^n$ for some integer $\lambda > 1$.

ex: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \overset{h}{\longmapsto} \begin{pmatrix} x_2 \\ x_1 + x_2^2 \end{pmatrix}$

$$\deg(h^n) = 2^n.$$

Q: What kind of sequences can we get by looking at

$$n \longmapsto \deg(f^n)$$

for $f \in \underset{\text{End}}{\text{Aut}}(\mathbb{A}^m_k)$, ?

ex: Dim $m = 3$

$$(x_D:) \quad x_1^2 + x_2^2 + x_3^2 = x_1 x_2 x_3 + \underset{\mathbb{R}}{\overset{\cap}{D}}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\sigma_3}{\longmapsto} \begin{pmatrix} x_1 \\ x_2 \\ x_3' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_1 x_2 - x_3 \end{pmatrix}$$

$$\deg\left((\sigma_2 \circ \sigma_3)^n\right) \sim n$$

$$\deg\left((\sigma_1 \circ \sigma_2 \circ \sigma_3)^n\right) \sim \lambda^n,$$

where $\lambda = \frac{1+\sqrt{5}}{2}$ golden ratio.

**Thm**: (C. Urech)
Let $h \in \mathrm{Aut}(\mathbb{A}^m_k)$.
If $n \mapsto \deg(h^n)$ is not bounded, then

$$\max_{j=0}^{n} \deg(h^j) \geq C_m \cdot n^{\frac{1}{m}}$$

Pf: (1) $\underbrace{\mathrm{End}_{\leq d}}_{k\text{-vector space}}(\mathbb{A}^m_k)$ = end. defined by formulas of deg $\leq d$.

$$\text{dimension} = m \times \dim\left(k[x_1, \dots, x_m]_{\leq d}\right) = m\binom{m+d}{m}.$$

"Choose" b/c

$$x_0^{i_0} x_1^{i_1} \cdots x_m^{i_m}, \quad i_0 + i_1 + \cdots + i_m = d$$

$$\underbrace{\overbrace{\_\ \_}^{x_0^2} \times \overbrace{\_\ \_}^{x_1^2} \times \overbrace{\_\ \_\ \_\ \_\ \_}^{x_2^4}}_{m+d}$$

$$m = 2, \quad d = 8$$

(2) $h \in \mathrm{End}(\mathbb{A}^m)$

Assume linear relation among the iterates:

$$h^5 = h^3 - 2h + \mathrm{Id}.$$

Compose the right w/ $h^n$

$$h^{5+n} = h^{3+n} - 2h^{1+n} + h^n$$

$$= \sum_{k=0}^{4} a_k h^k$$

$a_k \in$ Field $k$

= degree is unif bounded by $\max_{j=0}^{4} \deg h^j$.

(3) Put step (1) & (2) together:

$$D_h(n) = \max_{j=0}^{n} \deg(h^j)$$

$$n+1 \geq m \binom{m + D_h(n)}{m}$$

$$\Downarrow$$

bounded degrees $\square$

## 4. Finite subgrps

**4.A Prop** If $G$ is finite subgrp of $\mathrm{Aut}(\mathbb{A}^2_k)$,

then (1) $G$ is conj to a subgrp of $\mathrm{Aff}_2(k)$ or $E$

(2) if $\mathrm{char}(k) = 0$, then $G$ is __ $GL_2(k)$.

## 4.B Fixed pt thm.

**Thm** Let $G$ be subgrp of $\mathrm{Aut}(\mathbb{A}^m_k)$ st

(i) $G$ is a p-grp
$\#G = p^r$, $p$ prime, $r \geq 1$.

(ii) $\mathrm{char}(k) \neq p$ and
$k$ alg. closed.

Then $G$ has a fixed pt.

**Consequence** $G$ fixes $0$, the origin.

$$\Phi = \sum_{g \in G} \underbrace{(Dg)_0^{-1}}_{\substack{\cap \\ GL_m(k)}} \circ g$$

$\in \mathrm{End}(\mathbb{A}^m_k)$

$(D\Phi)_e = (\#G)\, \mathrm{Id} \in GL_m(k)$

$$\Phi \circ h = (Dh)_0 \circ \Phi, \ \forall h \in G.$$

**Corollary:** $G \ni g \longmapsto (Dg)_0 \in GL_m(k)$ is injective.

**Thm:** [Minkowsky] Set

$$\mathrm{Mink}(m,\ell) := \left\lfloor \frac{m}{\ell-1} \right\rfloor + \left\lfloor \frac{m}{\ell(\ell-1)} \right\rfloor + \cdots + \left\lfloor \frac{m}{\ell^j(\ell-1)} \right\rfloor + \cdots$$

(= finite sum.)

If $G$ is $p$-gp in $GL_m(\mathbb{Q})$, then $\#G = p^r$ satisfies

$$r \leq \mathrm{Mink}(m,p),$$

and this upper bound is optimal.

**Tmrw:** fixed pt theorem.

## Character Variety –

- $F_2$ – free group of rank 2 $= \langle a, b \rangle$
- $\mathrm{Rep}(\mathbb{F}_2, SL_2(k)) = SL_2(k) \times SL_2(k)$
- $\begin{pmatrix} \rho : F_2 \to SL_2 \\ \varphi \in \mathrm{Aut}(\mathbb{F}_2) \end{pmatrix} \longmapsto \rho \circ (\varphi^{-1})$

$\chi(\mathbb{F}_2, SL_2) = \mathrm{Rep} /\!/ $ conjugacy in $SL_2$.

$\mathrm{Out}(\mathbb{F}_2) = \mathrm{Aut}(\mathbb{F}_2) / \mathrm{Inner}(\mathbb{F}_2)$

$\chi = \mathbb{A}^3_k$. $\mathrm{Out}(\mathbb{F}_2) \curvearrowright \mathrm{Aut}(\mathbb{A}^3_k)$

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = \begin{matrix} \mathrm{tr}(A) \\ \mathrm{tr}(B) \\ \mathrm{tr}(AB) \end{matrix}$$

$$D = \mathrm{tr}[A, B] + 2$$

**Q2** : exp growth where $\lambda \neq$ deg:

Take $h = e_2 \cdot t \cdot e_1 \cdot e_2^{-1}$

$\deg h = \max(e_1, e_2) \cdot e_2$

Now iterate: $e_2 (t \cdot e_1)^n e_2^{-1}$

**Q3** : Analog Struct:

→ hard part was that A, E generate the grp (we <u>assumed</u> that).

→ After: easy, use the free prod $A \backslash S * E \backslash S$.