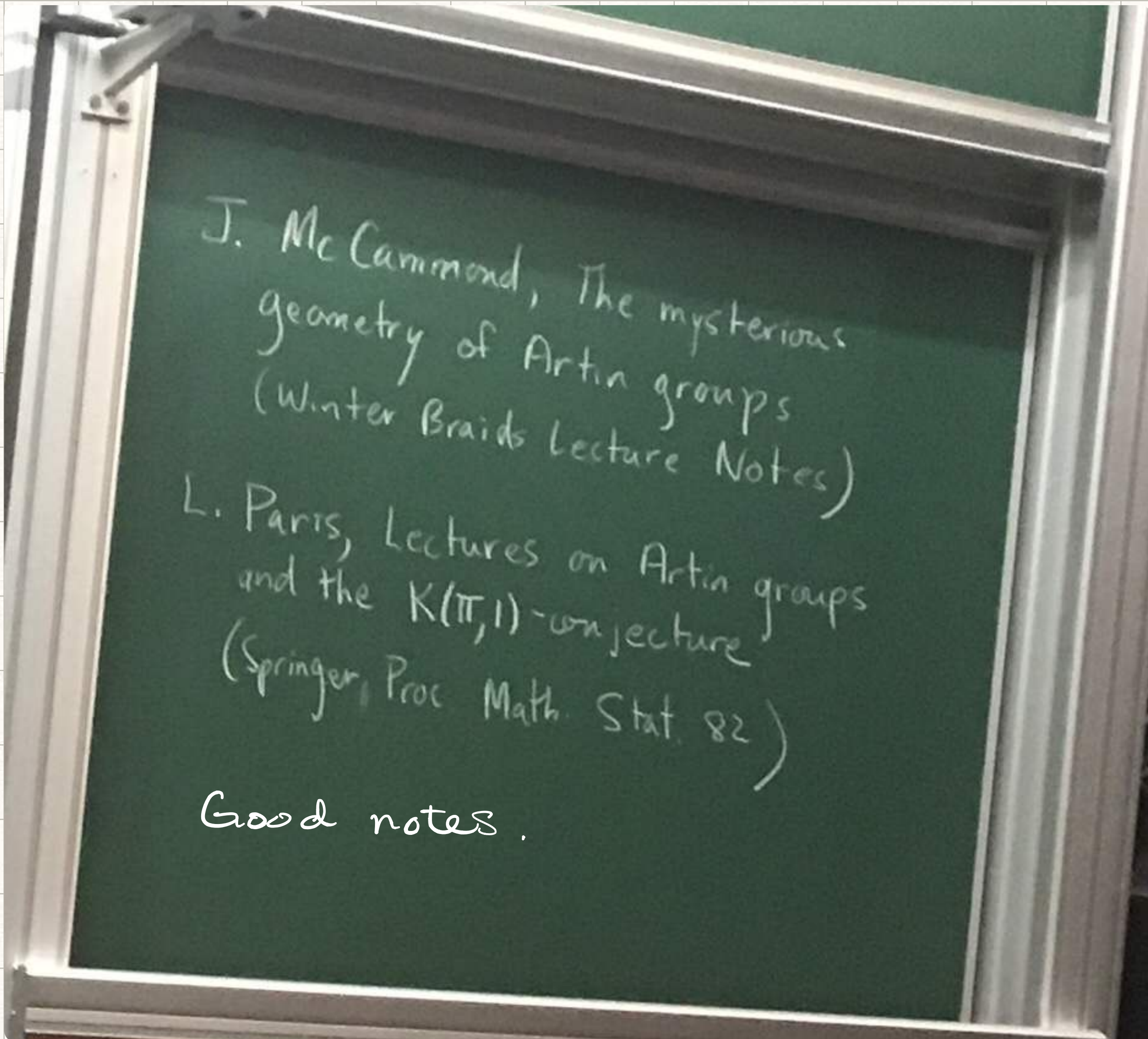


Artin-Tits ← he really came up w/ it.

What are Artin grps?

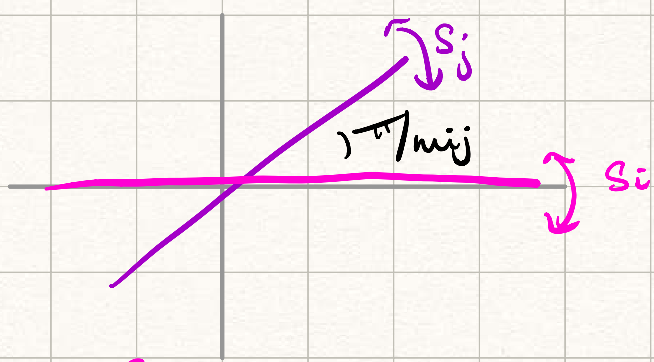
Start w/ Coxeter grps.

$W = \langle s_1, \dots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$

$m_{ij} \in \{2, 3, \dots, \infty\}, i \neq j$

↑ generalization of dihedral.

Geometrically: $W \curvearrowright \mathbb{R}^2$ discrete grp generalized by reflections.



two reflections = rotation

$m_{ij} = \infty$

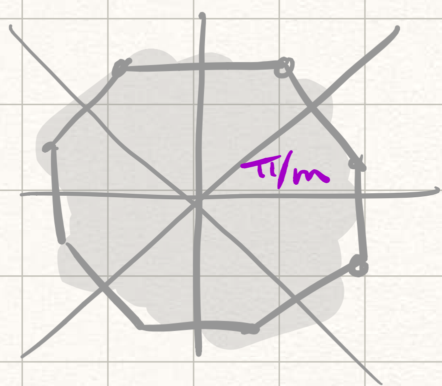
We encode this info in a labelled graph Γ.

$\begin{cases} \text{vertices}(\Gamma) = \{s_1, \dots, s_n\} \\ \text{edges}(\Gamma) : \quad s_i \overset{m_{ij}}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} s_j \quad \text{wherever } m_{ij} < \infty \end{cases}$

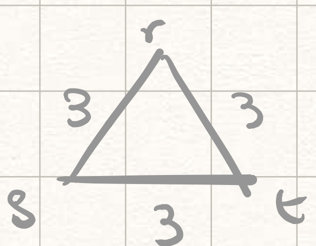
then W_Γ is the associated Cx. grp

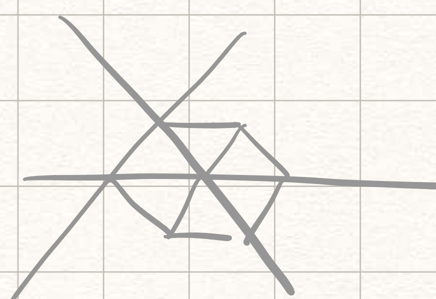
<u>ex</u>: $\Gamma = \; s \overset{m}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} t$

$\Rightarrow W_\Gamma = D_{2m}$ dihedral grp

$\Gamma =$ 

<u>ex</u>: $\Gamma = \; \underset{s}{\bullet} \quad \underset{t}{\bullet} \quad \Rightarrow W_\Gamma = D_\infty$

<u>ex</u>: 

$\rightarrow$ 

We tile the entire plane,
hence it is an ∞-grp.

Artin grp

$$A_\Gamma = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \dots}_{m_{ij} \text{ repeats}} = \underbrace{s_j s_i s_j \dots}_{m_{ij} \text{ repeats}} \rangle$$

N.B. if we had $s_i^2 = 1$,
we recover the Cox. grps.

Geometric descript

$$W_\Gamma \curvearrowright \mathbb{R}^n \quad \rightsquigarrow \quad W_\Gamma \curvearrowright \mathbb{C}^n$$

Each reflection $r \in W_\Gamma$ fixes
a complex hyperplane $H_r \subset \mathbb{C}^n$.

The associated "hyperplane complement"
is the space

$$\mathcal{H}_\Gamma = \mathbb{C}^n \setminus \bigcup_r H_r$$

need to
restrict for ∞.

Then W_Γ acts freely on $\mathcal{H}_\Gamma$ and
$A_\Gamma = \pi_1(\mathcal{H}_\Gamma / W_\Gamma)$.

ex: Classique ex

$W_\Gamma = \Sigma_n$ = symmetric grp on n letters.

$\Sigma_n \curvearrowright \mathbb{C}^n$ by permuting coord.

Reflections:

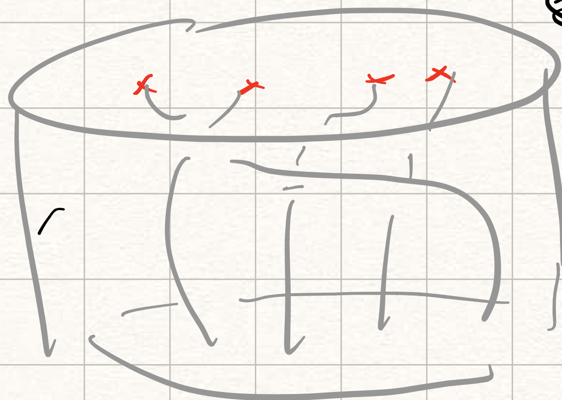
τ_{ij} = interchange ith & jth coordinate.

$H_{ij} = \{(z_1,\dots,z_n) \mid z_i = z_j\}$.

$\mathcal{H}_\Gamma = \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid z_i \neq z_j, \ \forall i \neq j\}$

= configuration of pts of n pts in $\mathbb{C}$.

$A_\Gamma = \pi_1(\mathcal{H}_\Gamma / W_\Gamma)$ = braid grps of small strands.



Artin Γ come in two flavours

- finite type Artin ($\iff W_\Gamma$ finite)

  no one knows [illegible]

- infinite type ($\iff W_\Gamma$ infinite).

- We know a great deal about Γ, and V little about about infinite type.

→ Finite type A_Γ has a "Garside Struct", which gives norm. form, bi-ant struct.

- In fin. types A_Γ - no such tool, the effective techniques are geom.

Old Conjs (#1)

1) A_Γ has solvable WP.
2) A_Γ tors free
3) If A_Γ is irred, $(A_\Gamma \neq A_{\Gamma_1} * A_{\Gamma_2})$
~~then~~

$$\text{center}(A_\Gamma) = \begin{cases} \mathbb{Z} & \text{if } A_\Gamma \text{ fin type} \\ \{1\} & \text{-- -- infinite} \end{cases}$$

4) A_Γ has a finite $K(\pi,1)$ space (namely Salvetti complex for A_Γ).

$$\boxed{\begin{array}{l} \text{5) } \mathcal{H}_\Gamma/A_\Gamma \text{ is a } K(A_\Gamma, 1) \text{ space} \\ \iff \widetilde{\mathcal{H}}_\Gamma \text{ contractible.} \end{array}}$$

called $K(\pi,1)$ conj for A_Γ

Early 70s: Deligne, Brieskorn-Saito proved that all these conj hold for finite type A_Γ.

New questions

1) Q : Is A_Γ hyp ?

A : Yes $\Leftrightarrow$ A_Γ has no edges

etc. (A_Γ is free).

2) Is A_Γ CAT(0)? $\underbrace{\text{Is } A_\Gamma \text{ CAT(0)?}}_{\text{A: don't know}}$ $\underbrace{\overset{\text{co-cmpt}}{\text{Cubically}} \text{ CAT(0)?}}_{\text{A: often no.}}$

! We don't know how to construct CAT(0) spaces which are not cubical.

3) Is A_Γ/center acyl. hyp?

Hoping for yes.

4) Hierarchically hyp.

Curve complex analog.

Terminology: Suppose A_Γ is an Artin grp w/ gen set: $S = \mathrm{Vert}(\Gamma)$,

For any $T \subseteq S$,
let A_T = subgrp of A_Γ gen by T.

Fact: If Δ_T = subgraph of Γ spanned by T.
Then $A_T \cong A_{\Delta_T}$.

These are known as special subgrps their conj are called parabolic subgrp.

1990's: Tried to make a cube complex which retracts to $\tilde{H}_\Gamma$.
(called Deligne complex),
ie $D_\Gamma \simeq \tilde{H}_\Gamma$.

Now, suffices to show D_Γ to show $\tilde{H}_\Gamma$ contractible.

⇝ only need to show it's CAT(0), since every CAT(0) space is contractible.

→ only need npccc & simply connected.

D_Γ : vertices $= \left\{ aA_T \mid a \in A_\Gamma, T \subseteq S, \; A_T \text{ finite type} \right\}$

edges: aA_T —— $aA_{T \cup \{s_i\}}$

cubes: intervals

$[aA_T, aA_R]$

$aA_T \subset aA_R$