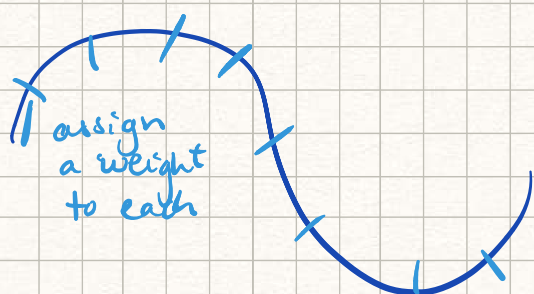
Non-pos curvature

I) CAT(0) inequality

Def: X is geodesic if every 2 pts is joined by geodesic (NOT necessarily unique)

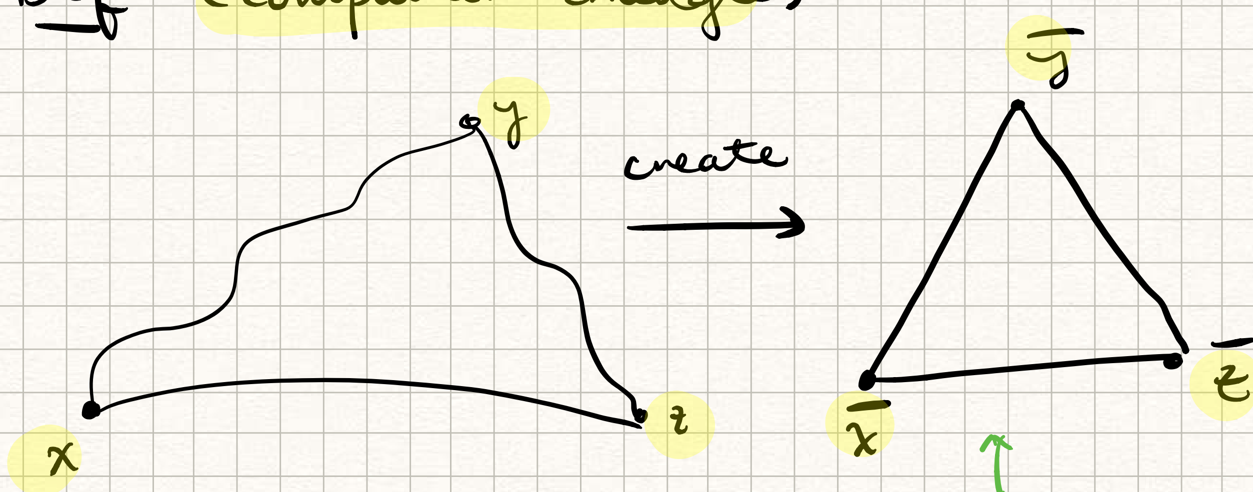
Q: what do we really mean by geodesic?

A:



then take the sup of every partition.

Def (comparison triangle)



comparison triangle

st
$$d(x,y) = d(\bar{x},\bar{y})$$
$$d(y,z) = d(\bar{y},\bar{z})$$
$$d(z,x) = d(\bar{z},\bar{x})$$

Now create comparison map $p \to \bar{p}$ which sends isometrically

$[x, y] \to [\bar{x}, \bar{y}]$

$[y, z] \to [\bar{y}, \bar{z}]$

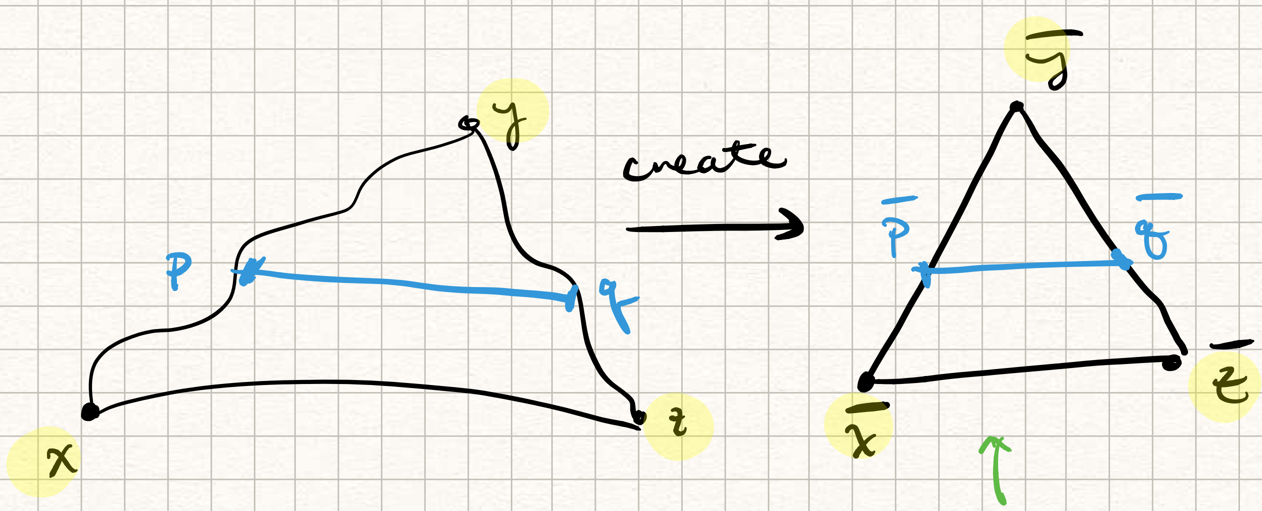
$[z, x] \to [\bar{z}, \bar{x}]$

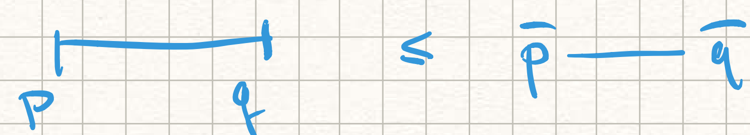
Def: The triangle $[x, y, z]$ satisfies the CAT(0) inequality

if $\forall p \in [x, y]$

$\forall q \in [y, z]$

$$d_X(p, q) \leq \underbrace{d_{E^2}}_{\text{Euclidean}}(\bar{p}, \bar{q})$$





Culture: CAT = Cartan
Alexander
Topamogov

Q: What is the diff b/w hyperbolic & CAT(0)
A: In CAT(0):
- every pair of pts is joined by a unique geodesic
- E^2 is CAT(0)

First examples

- n-dimensional Euclidean space $\mathbb{E}^n$
- $\mathbb{H}^2$
- $\underbrace{S^2}_{\text{unit sphere}}$ is NOT CAT(0).

Think of CAT(0) as curvature at most 0.

- universal cover of Riemannian manifold w/ sectional curvature $K \leq 0$.
- trees!

So again, ∃ curvature at most −1

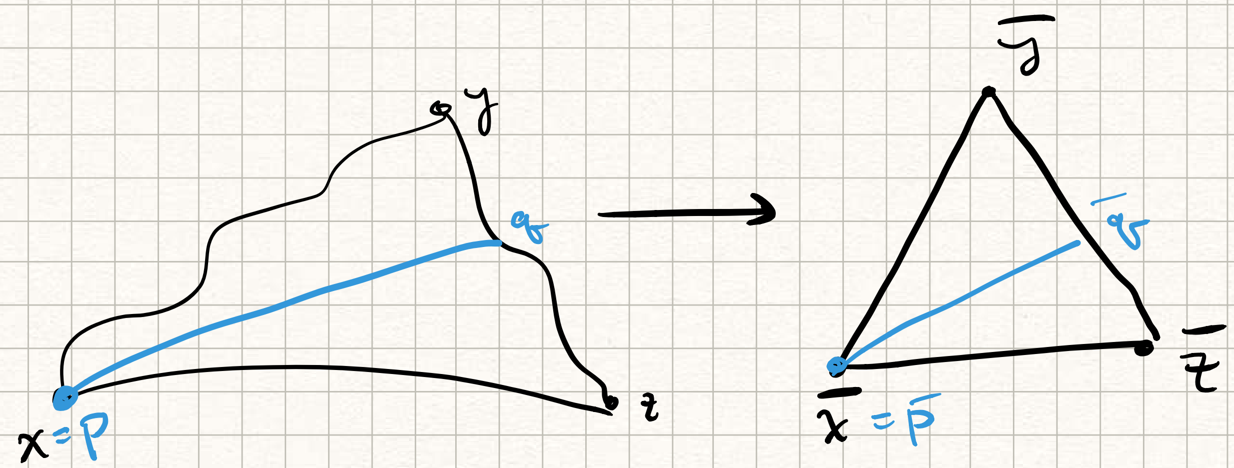
Rmk: Similar notion of CAT(−1) spaces, compare w/ triangles $\underbrace{\mathbb{H}^2}_{\text{hyperbolic }\Delta}$.

CAT(1) — $\mathbb{S}^2$.

⇝ where perimeter $< 2\pi$.

Useful Reformulation of CAT(0) ineq

i) It is sufficient to decide the CAT(0) inequality for every geodesic $[x, y, z]$ w/ $p = x$ FIXED and $q \in [y, z]$.



ii)

X is CAT(0) $\iff$

- X is geodesic (and complete)
- $\forall\, x, y, z \in X$ if m is the midpt of $[y, z]$

& $d(x,y)^2 + d(x,z)^2 \geq 2 d(x,m)^2 + \frac{1}{2} d(y,z)^2$.

II) Buiding New CAT(0) Spaces

A) Direct product

Given (X_1, d_1), (X_2, d_2) two metric spaces,

let $X = X_1 \times X_2$

be defined as:

if $x = (x_1, x_2)$
$y = (y_1, y_2)$

set $d(x,y) = \sqrt{d(x_1,y_1)^2 + d(x_2,y_2)^2}$

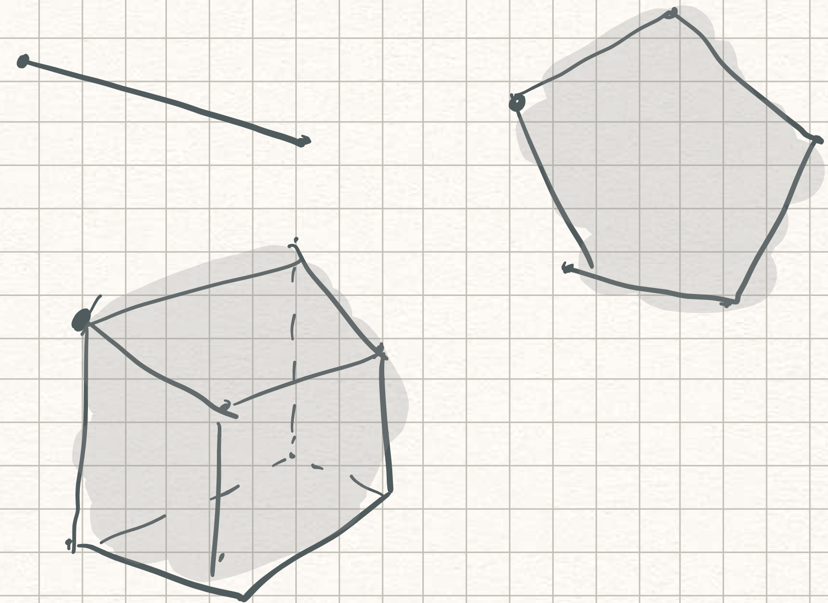
Fact: If X_1, X_2 CAT(0) then X is.

ex:
- $\mathbb{R} \times \mathbb{R} = \mathbb{E}^2$
- $\mathbb{E}^n \times \mathbb{E}^m = \mathbb{E}^{n+m}$
- tree $\times$ tree

B) Polyhedral complexes

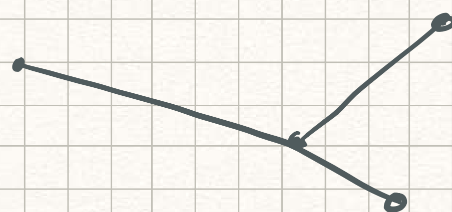
Def: convex polyhedron is convex hull of finitely many pts in $\mathbb{E}^n$.

ex :



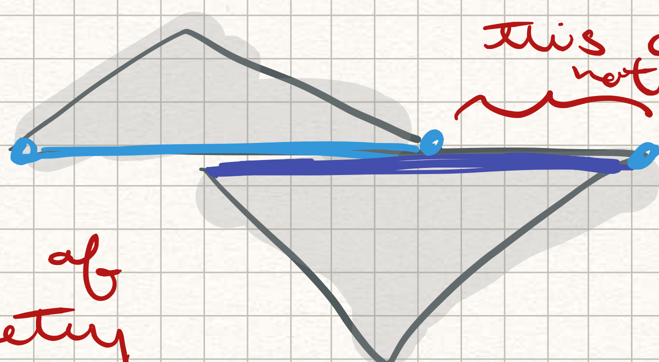
polyhydral complex : space by gluing polyhedron isometrically along their faces.

ex :



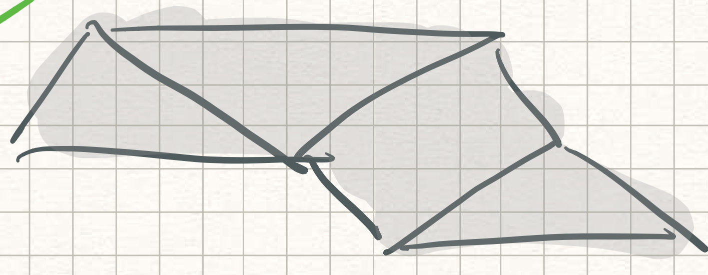
graph

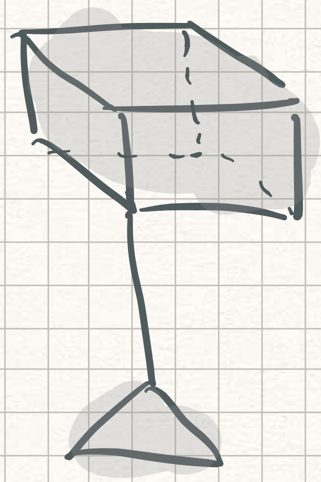
NOT



this does not have isom.

lack of isometry





→ poly complex is geodesic!

→ ∃ simple condition for poly complex to be CAT(0).