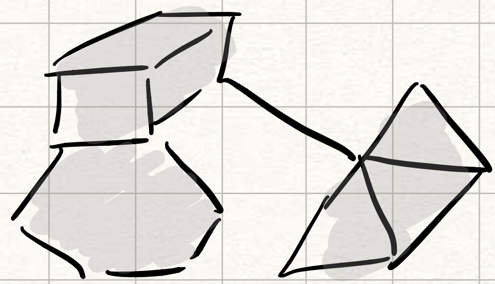
Polyhedral complexes

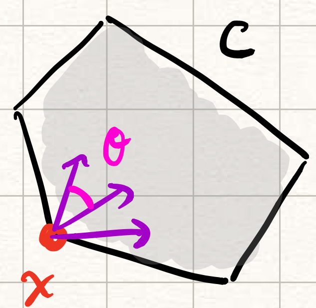


Def:
The shapes are the isometric classes of faces.

Thm (Bridson)
A polyhedral cmplx w/ finitely many shapes is geodesic and complete.

Q: When are poly cplx's CAT(0)?
- simply connected
- local cond. (links).

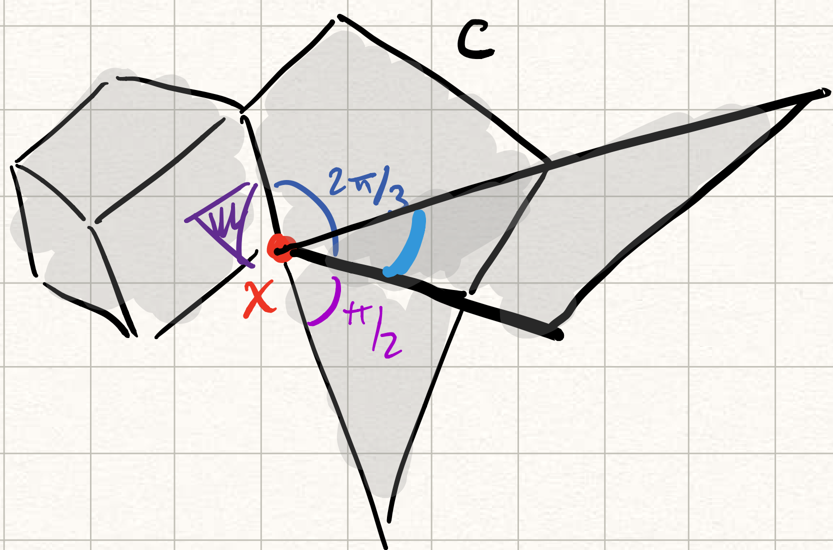
Link of a pt



θ = distance b/w 2 directions

$lk(x, C)$ = set of directions from x w/ angle metric.

$lk(x, X)$ = "attach together" $lk(x, C)$



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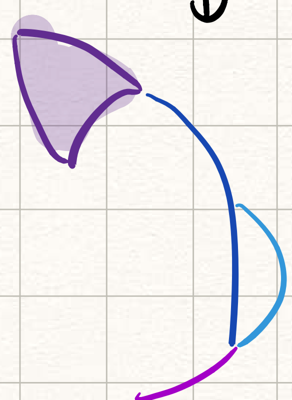


& use metric from pieces.

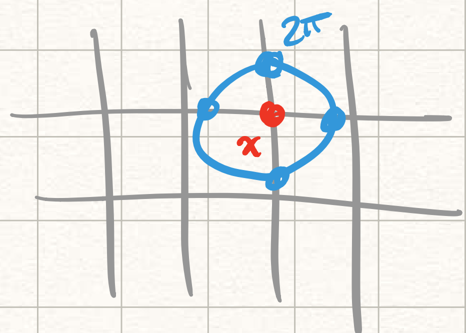
Thm: Assume that X is a polyhedral cmplx w/ finitely many shapes

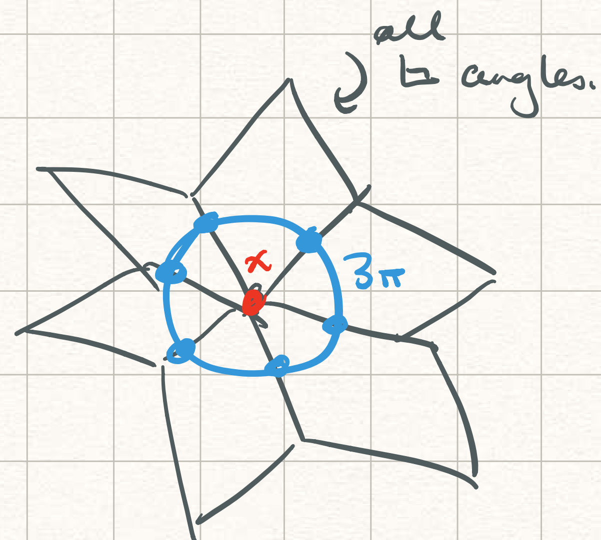
Then X is CAT(0) iff
- X is simply connected
- the link of every pt is CAT(1)

→ this is HARD to check.

easy cases:
- if X is a graph ($\Leftrightarrow$ X is tree).
- if X is 2-dim,
  link = graphs
  link: condition $\Leftrightarrow$
  every loop in the link has perimeter $\geq 2\pi$.

ex:





exc: If X is a cube complex, then the link condition is equivalent to "flag cond"

PT: link cond $\Leftrightarrow$ flag cond.

From now on:

X is always a CAT(0) space.

I. Geodesics

- geodesics are unique

Lemma: If $c : [a,b] \to X$ is locally a geodesic path, then c is geodesic.

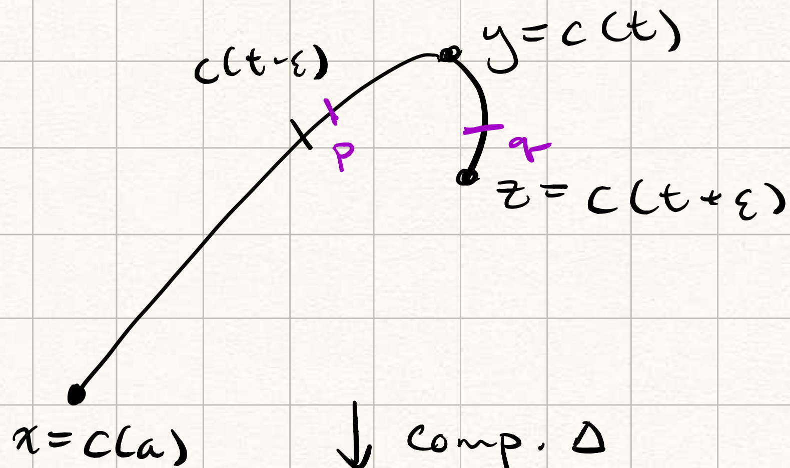
Def (locally geod)

$\forall t \in [a,b]$, $\exists \varepsilon > 0$ st $c|_{[t-\varepsilon, t+\varepsilon]}$ is geodesic.

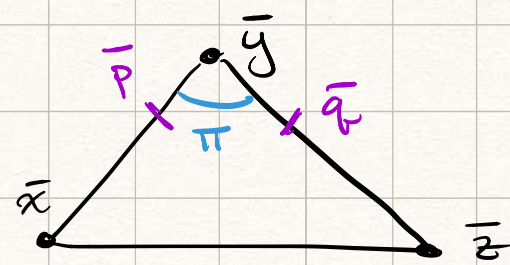
Pf: $I = \{ t \in [a,b] : c|_{[a,t]} \text{ is geod} \}$

- I is non-empty, closed
- I is open.

Fix $t \in I$, $c|_{[t-\varepsilon, t+\varepsilon]}$ geod.



$\downarrow$ comp. Δ



$$
\begin{aligned}
d(x,y) + d(y,q) &= d(p,q) \\
&\le d(\bar p, \bar q) \\
&\le d(\bar p, \bar y) + d(\bar y, \bar q) \\
&= \underline{d(p,y) + d(y,q)}
\end{aligned}
$$

$\Rightarrow \measuredangle = \pi$.

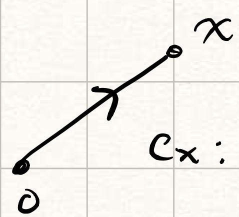
got lost.

$d(x,y) + d(y,z) = d(x,z)$

$\Rightarrow$ geodesic

Prop: A CAT(0) space is contractible.

Fix a pt $o \in X$, $x \in X$, c_x geod.



$c_x : [0, d(o,x)] \to X$

homotopy map.

Let $H : [0,1] \times X \to X$

$(t, x) \to c_x(t\, d(o,x))$

$H|_{\{0\} \times X} = o \quad , \quad H|_{\{1\} \times X} = \mathrm{id}$

exc: H is continuous.

(need to use comp. Δ).

Convexity

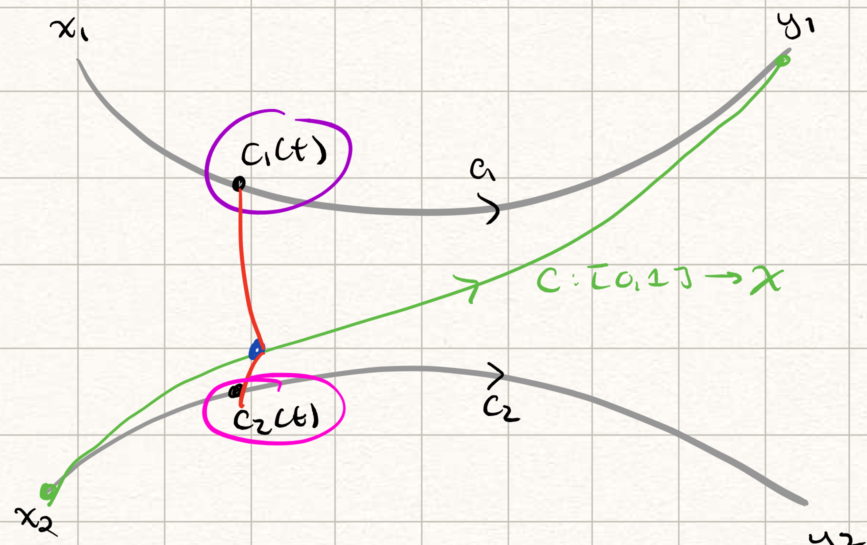
A fn $f: I \to \mathbb{R}$ is convex if $\forall a, b \in I$,
$\forall t \in [a,1]$,

$$f(ta + t(1-t)b) \leq tf(a) + (1-t)f(b)$$

Prop: Assume that $c_1, c_2 : [0,1] \to X$ geodesics param w/ ct speed.

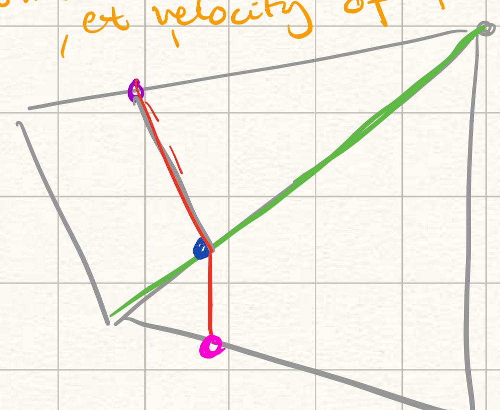
then $t \to d(c_1(t), c_2(t))$ is convex.

X:



$\mathbb{E}^2$:

sim Δ due to ct velocity of param c.



$\Rightarrow d(\bar{c}_1(t), \bar{c}(t)) = t\, d(\bar{x}_1, \bar{x}_2).$

$d(\bar{c}_1(t), \bar{c}_2(t)) = (1-t)(\bar{y}_1, \bar{y}_2)$

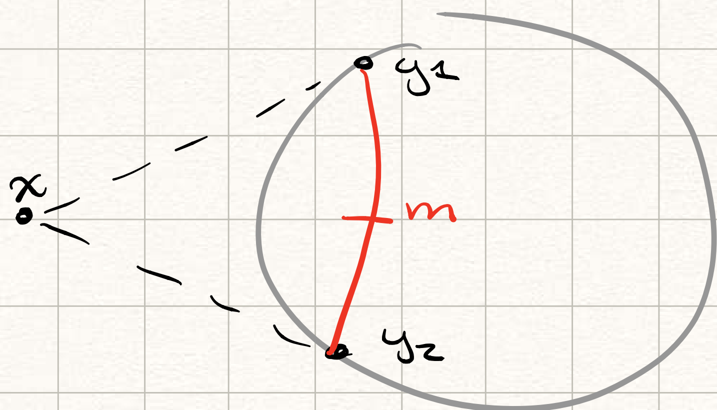
$\Rightarrow d(c_1(t), c_2(t)) \leq t\, d(x_1, x_2) + (1-t)\, d(y_1, y_2).$

Def: $Y \subset X$ is convex so that $\forall y, y' \in Y$
$\Rightarrow [y, y'] \subset Y$.

(e.g. geods, balls, isometrically emb'd flat)
$\mathbb{E}^n$

Prop: X is CAT(0), $Y \subset X$ convex & complete
$\forall x \in X$, $\exists!\, y \in Y$ st $d(x,y) = d(x, Y)$.
y is the projection of x onto Y.

Pf: Assume not.



Uniqueness

$$\tfrac{1}{2} d(y_1, y_2)^2 \leq d(x, y_1)^2 + d(x, y_2)^2 - 2 d(x, m)^2$$
$$\leq d(x, y_1)^2 + d(x, y_2)^2 - 2 d(x, Y)^2$$
$$= 0$$

existence

$y_n \in Y$ st $\quad d(x, y_n) \to d(x, Y) \quad (\star)$

$$\frac{1}{2} d(y_n, y_m)^2 \le d(x, y_n) + d(x, y_m)^2$$
$$\le d(x, y_n)^2 + d(x, y_m)^2 \longrightarrow 0$$

$\Rightarrow$ (y_n) is Cauchy

$\Rightarrow$ it converges to $y \in Y$ one checks $d(x, y) = d(x, Y)$

$\star$: Why can we have this seq?

A: $d(x, Y) := \inf\limits_{y \in Y} d(x, y)$.