Boundary at ∞ of CAT(0) space

1) Boundary as a set

def: geodesic ray: $c : \mathbb{R}_+ \to X$ geod.

want to say that any two parallel rays $\to$ same pt at ∞.

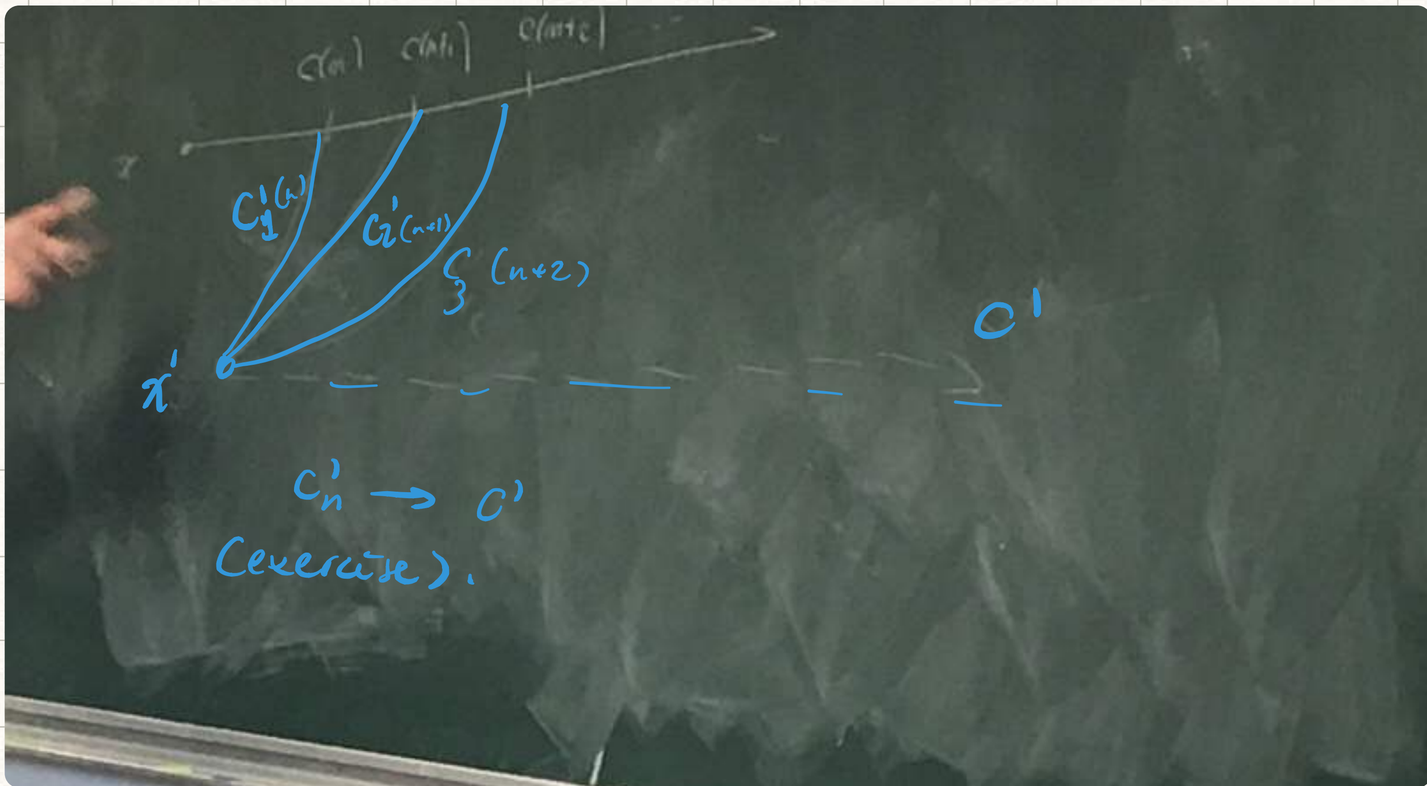
def: $c_1, c_2 : \mathbb{R}_+ \to X$ are asymptotic if $\exists K$ st $\forall t \geq 0$, $d(c_1(t), c_2(t)) \leq K$
we write $c_1 \sim c_2$.

The boundary at ∞

$\partial X = \{\text{geodesic ray}\}/\sim$
$= \partial_\infty X$.

Fact: fix $c : \mathbb{R}_+ \to X$ geod ray starting at x,
choose x'.
Then $\exists !$ geodesic ray c' starting at x'
st $c \sim c'$.

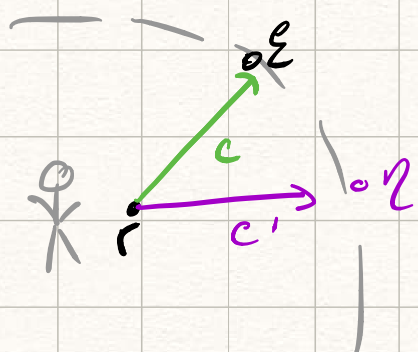
just one!
that is asympt!



exc: Show indeed that $c' = \lim c'_n(t)$
is indeed asymp to c
(& lim exists).

II - Topology on ∂X

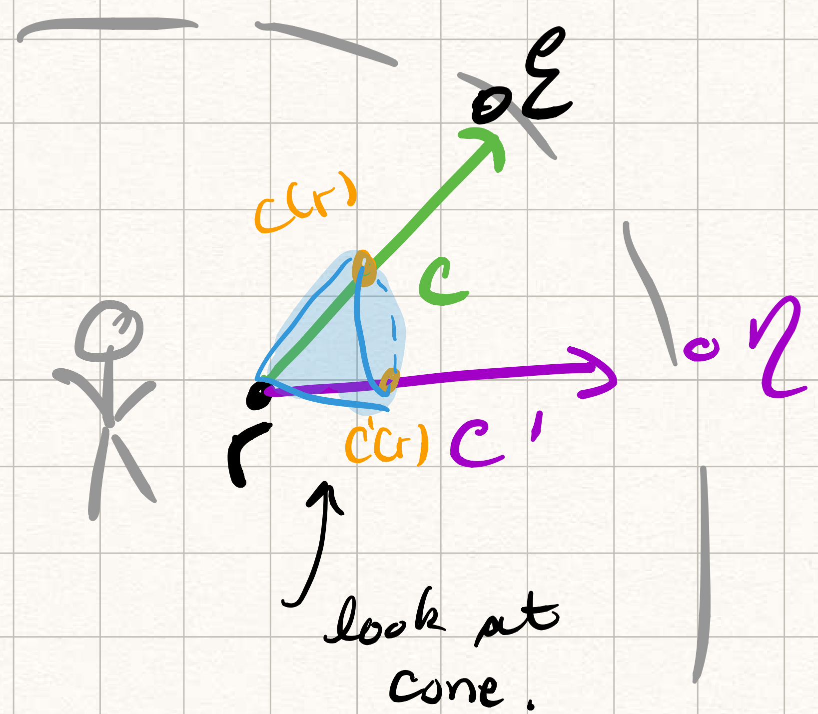
Fix $\forall$ a base pt $r \in X$



$\xi \in \partial X$ given by geod ray starting at O.

$\forall r > 0,\ \forall \varepsilon > 0,$

top : $U_\xi(r, \varepsilon) = \{\eta \in [c'] \in \partial X \mid d(c'(r), c(r)) < \varepsilon\}$



Claim: $\{U_\xi(r, \varepsilon)\}$ is a basis of open nbhd of ξ
⇝ called CONE TOPOLOGY

Q(naive) : why can't you define open interval in ∂X?
e.g. S^1, disk model.

A(me): Need to figure out what the pts are even.

e.g. • if X bounded, the $\partial X = \emptyset$
(no geod ray).

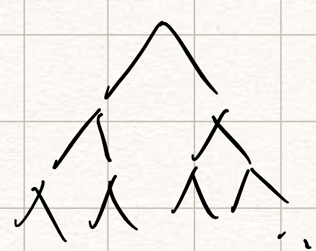
• $\partial \mathbb{R} = 2$ pts $\quad \xrightarrow[\mathbb{R}^+]{} \quad \xleftarrow[\mathbb{R}^-]{}$.
discrete top.

• $\partial \mathbb{E}^2 = S^1$, $\partial \mathbb{E}^n = S^{n-1}$

• X is Hilbert $\partial X = S(0,1)$

- $\partial \mathbb{H}^2 = S^1$, $\partial \mathbb{H}^n = S^{n-1}$.

• X is a 3 valent regular tree
$\partial X =$ Cantor set



• $\exists$ CAT(0) space st ∂X is
- a Menger curve
- Serpinski curve.

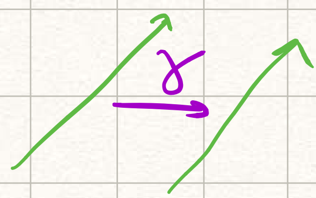
Properties

- If X is proper (closed ball $=$ compact), then ∂X is compact.

- If γ is an isometry of X, then γ induces an homeomorphism of ∂X.

- If X is proper and γ is parabolic isometry then γ fixes a pt in ∂X.

Note: translation fix everything



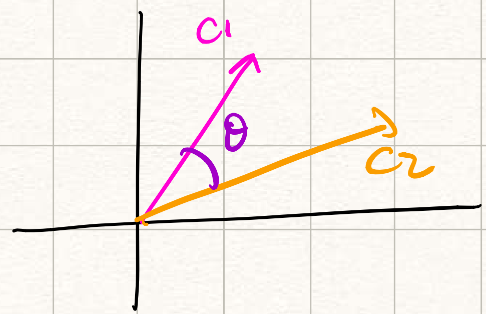
$\rightsquigarrow$ same boundary pt.

WARNING: if $f: X \to Y$ qi of CAT(0) sp, then in general, there is NO RELATION b/w ∂X and ∂Y.

CAT(0) spaces behave badly under quasi-isometries.

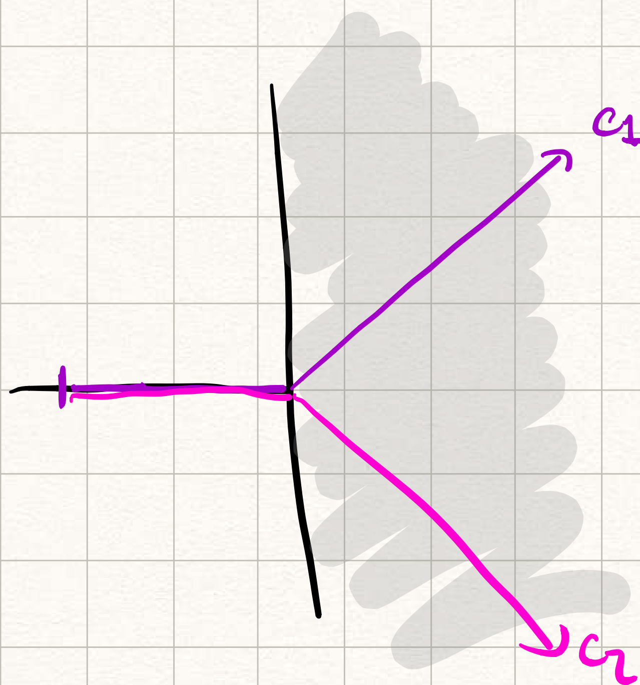
Metric on ∂X

i) $\mathbb{E}^3$



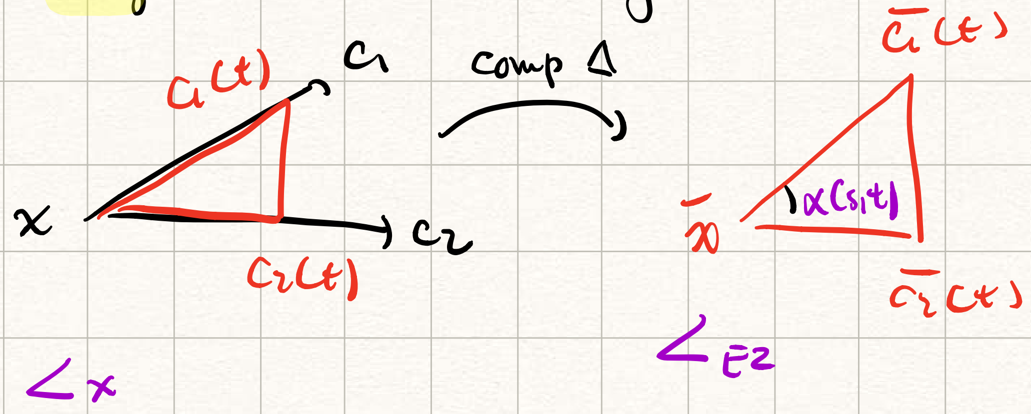
the intuitive dist b/w geod ray is angle!

ii)



here the $\theta = 0$, but want same dist b/w c_1, c_2.

Def: Angle b/w c_1, c_2 geods.

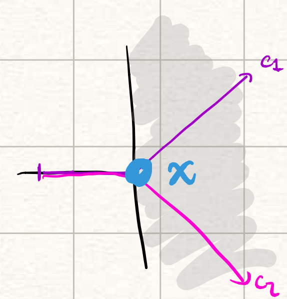


$$\angle_x(c_1, c_2) = \lim_{s,t \to 0} \alpha(s,t)$$

but now the bpt matters
$\rightsquigarrow$ we don't want it to.

Def: If $\gamma_1, \gamma_2 \in \partial X$, then the **angle** b/w γ_1, γ_2 is

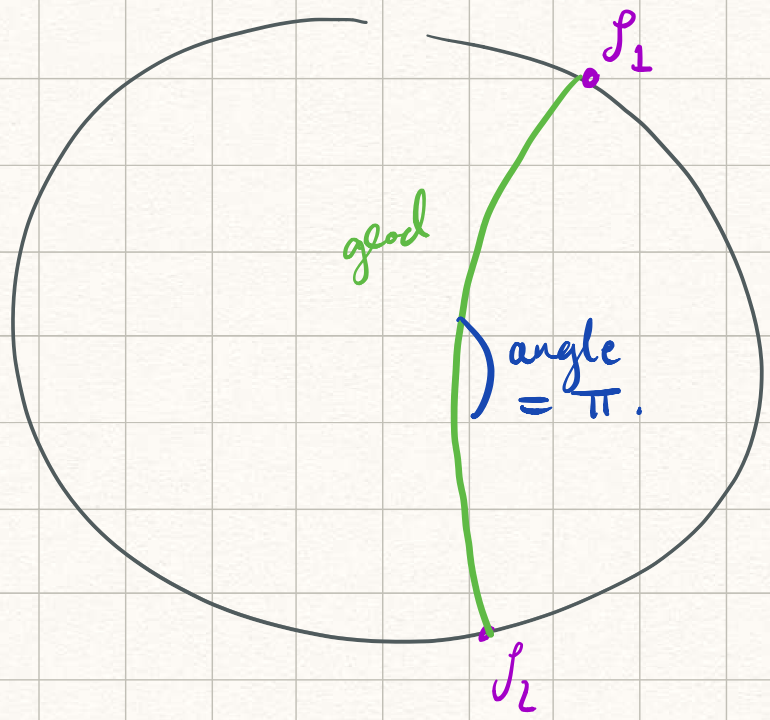
$$\angle(\gamma_1, \gamma_2) = \sup_{x \in X} \angle_x(\gamma_1(x), \gamma_2(x))$$



Prop: $(\partial X, \angle)$ is a complete metric space.

ex: $(\partial \mathbb{E}^2, \angle) = S^1$ w/ angle metric.

- $(\partial \mathbb{H}^2, \angle) = S^1$, $d(\xi_1, \xi_2) = \pi \quad \forall \xi_1 \neq \xi_2$



BUG $\rightarrow$ these two spaces are NOT homeo. $\leftarrow$ topo not what we defined!

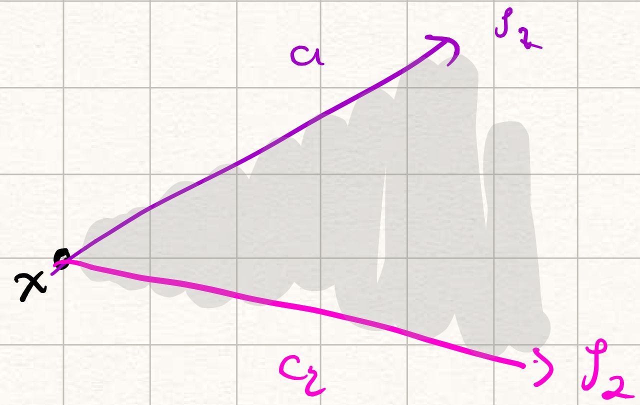
<u>Fact</u>: the map

$$\mathrm{id}: (\partial X, \angle) \longrightarrow (\partial X, \text{cone top})$$

is continuous.

$\Rightarrow$ our top is finer than cone top.

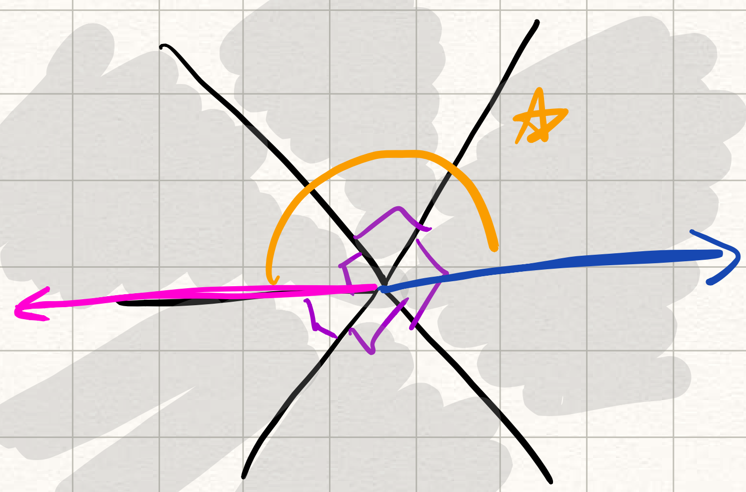
Fact: Take 2 pts, $\xi_1, \xi_2 \in \partial X$, $x \in X$,
st $\angle(\xi_1, \xi_2) = \angle_x(\xi_1, \xi_2) < \pi$



convex hull of c_1, c_2
is isometric to a sector
of $\mathbb{E}^2$.

IV - Tits Metrics

$\partial \mathbb{E}^2 = S^1$ w/ the angle metric.

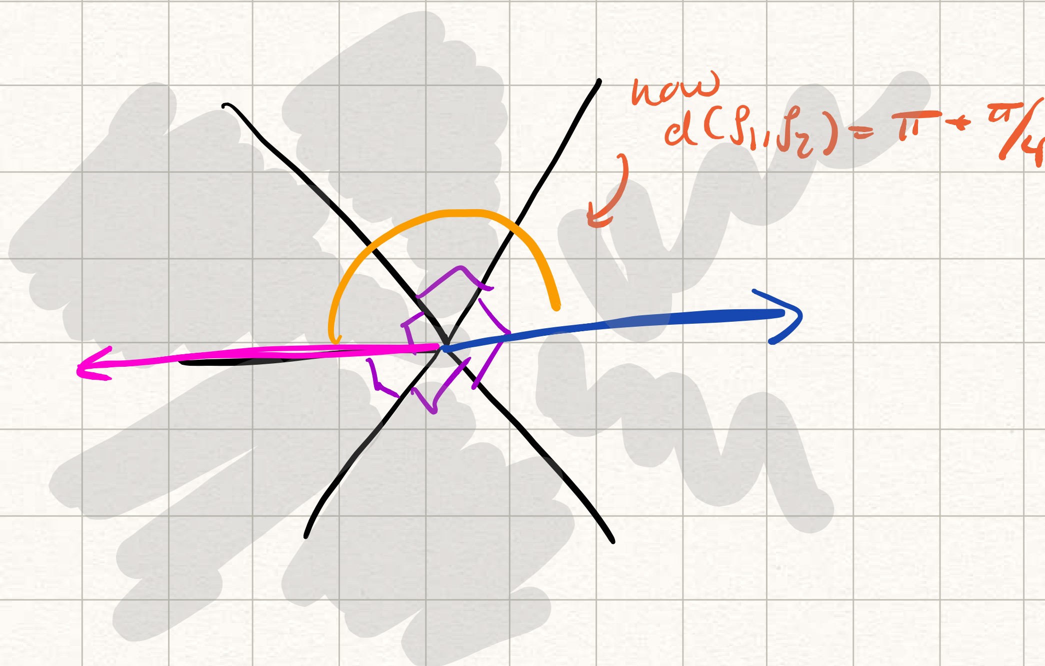


$\partial X = S^1_{\pi/2}$.

☆: Right now, $\angle$ is π, would like to fix that.

Def (Tits metric)

on ∂X is the len. metric on ∂X induced by $\angle$



exc: $X = X_1 \times X_2$

∂X.

nice prop → if can split ∂X, then X is direct prod.