Isometries of CAT(0) spaces

geometry of X $\longleftrightarrow$ Isom(X)

$\updownarrow$ $\quad$ $\updownarrow$

lattices in Isom(X)

Thm: (Flat torus)

Let X be a CAT(0) space, $\Gamma \curvearrowright X$ geometrically.

If Γ contains $A \cong \mathbb{Z}^n$ then $\exists$ convex, A-invar. subsp $Y \cong \mathbb{E}^n$.

consequence

by isometry

If $X \curvearrowleft \Gamma$ geometrically on a X CAT(0), every virtually solvable subgrp Γ is virtually abelian (in part, no $BS(1,2)$).

I. Translation len & min sets

X CAT(0), complete.

$\gamma \in \mathrm{Isom}(X)$

translation len

$$\tau(\gamma) = \inf_{x \in X} d(x, \gamma x)$$

min set

$$\mathrm{Min}(\gamma) = \{x \in X \mid d(x, \gamma x) = \tau(\gamma)\}$$

	$\mathrm{Min}(\gamma) \neq \emptyset$	$\mathrm{Min}(\gamma) = \emptyset$
$\tau(\gamma) = 0$	elliptic	parabolic
$\tau(\gamma) > 0$	hyperbolic	

exc: $X = \ell^2(\mathbb{N})$ (sq-Σ'able seqs)

$\gamma x = (1, x_1, x_2 \ldots)$

Show γ is parabolic.

Facts:

1) If $\Gamma \curvearrowright X$ geometrically, then every element in Γ is semi-simple.

2) X_1, X_2 CAT(0), set $X = X_1 \times X_2$

Take $\gamma_1 \in \mathrm{Isom}(X_1)$,
$\gamma_2 \in \mathrm{Isom}(X_2)$,

define $\gamma \in \mathrm{Isom}(X)$,
$\gamma(x_1, x_2) = (\gamma x_1, \gamma x_2)$

when is γ semi-simple?
Describe $\mathrm{Min}(\gamma)$.

Props: $\gamma \in \mathrm{Isom}(X)$

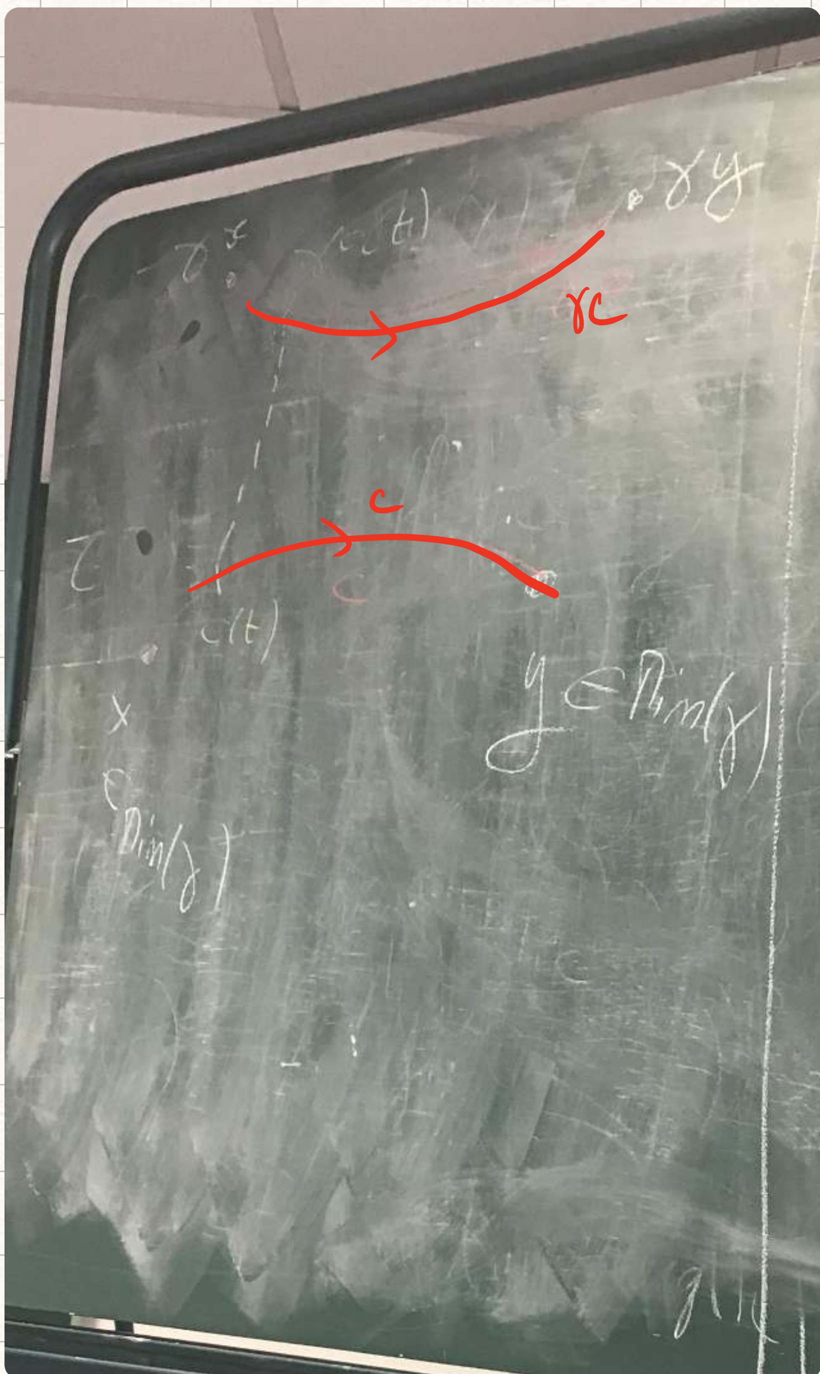
(1) If $u \in \mathrm{Isom}(X)$
$t(u\gamma u^{-1}) = t(\gamma)$
$\mathrm{Min}(u\gamma u^{-1}) = u\,\mathrm{Min}(\gamma)$

(2) $\mathrm{Min}(\gamma)$ is convex

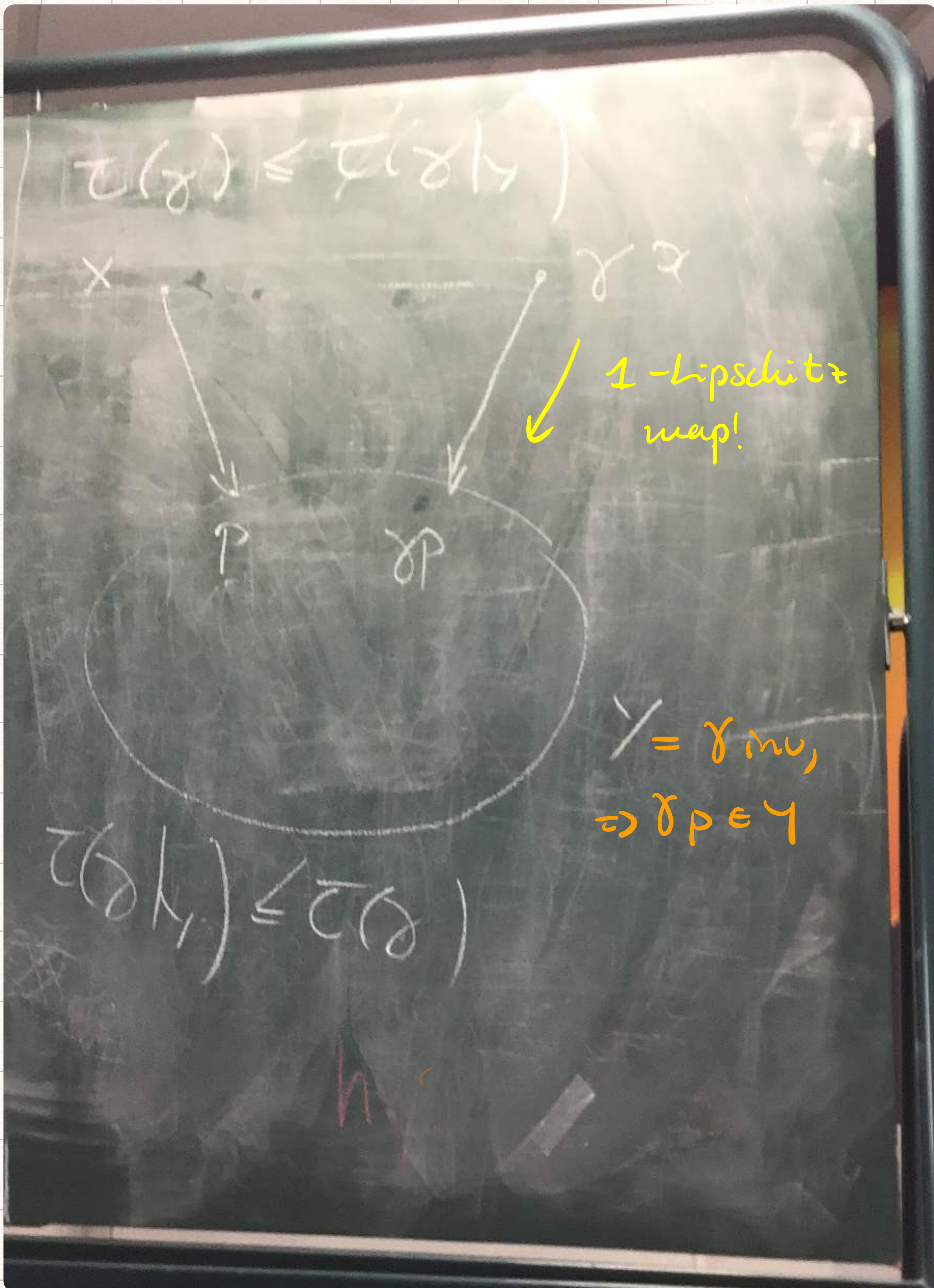
(3) If $Y \subset X$ is a convex subset, then $\gamma|_Y$ is semi-simple $\iff$ $\tau(\gamma|_Y) = \tau(\gamma)$

$$\mathrm{Min}(\gamma|_Y) = Y \cap \mathrm{Min}(\gamma)$$

<u>Pf (2)</u>



Pf 3

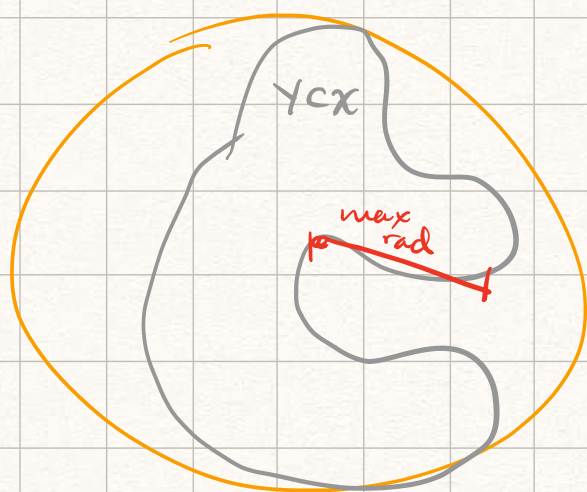


Elliptic isometries

Thm: If $\Gamma \curvearrowright X$ has a bounded orbit, then Γ fixes a pt in X.

→ finite order aut. are elliptic.
→ maximal cpct subgrp of $SL(n, \mathbb{Q}_p)$

Pf:



exc: Assume that $Y \subset X$ is bounded

$\forall x \in X, \quad r(x) = \sup_{y \in Y} d(x, y)$

$\exists! \, x_0 \in X$ st $r(x_0) = \inf_{x \in X} r(x)$

"x_0 circumcenter of Y"

Appl: if Γx is bounded, Γ fixes the circumcenter of Γx.

Fact: If $\Gamma \curvearrowright X$ geometrically, then up to conjugacy, Γ contains only finitely many finite subgrps.

Hyp isom

Def: $c: \mathbb{R} \to X$ is an axis of γ if c geod, $\exists a > 0$ st $\gamma(c(t)) = c(t+a)$ $\forall t$.

Obs: if $\tau(\gamma) = a$, then $c \subset Min(\gamma)$

"$\subset$ contained in"

ex: γ = transl. in $\mathbb{E}^2$
$Min(\gamma) = \mathbb{E}^2$.

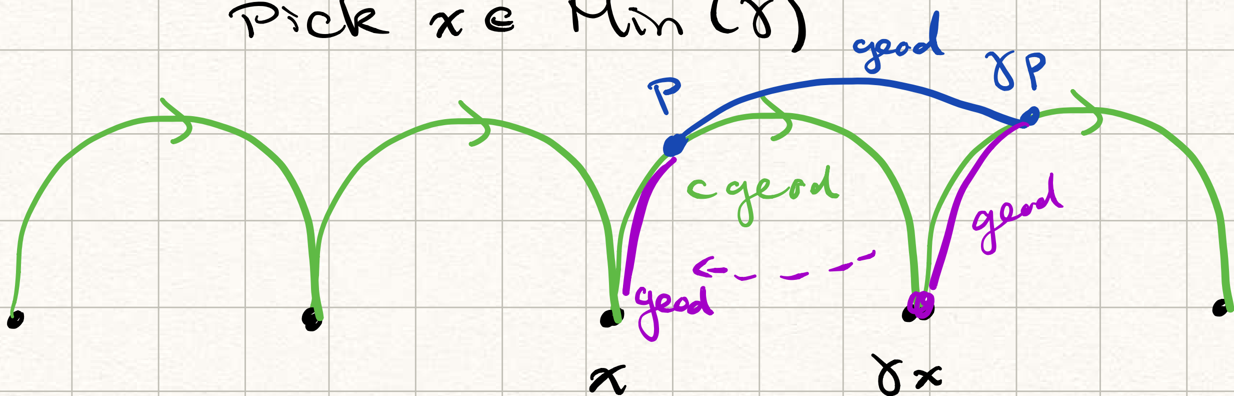
Prop: Assume γ is hyp, then $Min(\gamma) \equiv Y \times \mathbb{R}$ where Y is convex

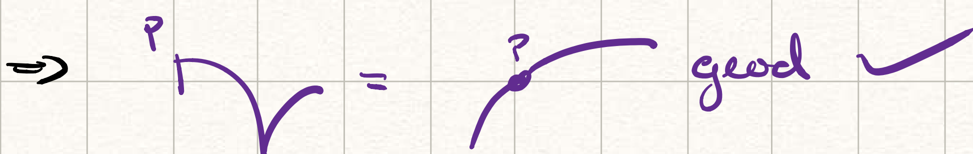
- $\gamma(y, t) = (y, t + \tau(\gamma))$

- If u centralizes γ, then it preserves Min C

Pf: 1) there is always an axis for hyp. isom

Pick $x \in \mathrm{Min}(\gamma)$



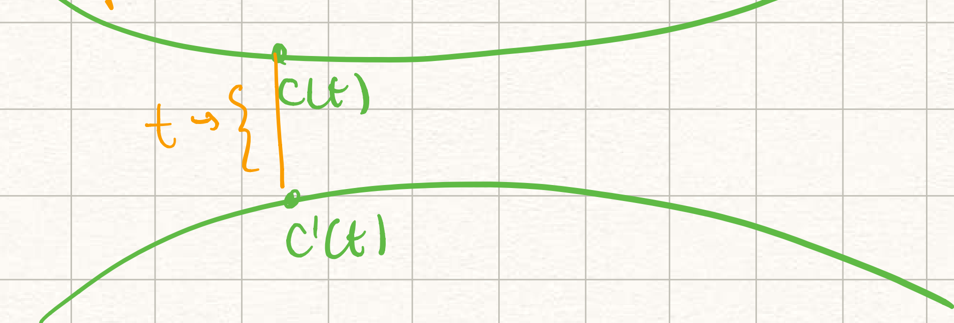


$p \in \mathrm{Min}(\gamma)$, $d(p, \gamma p) = \tau(\gamma)$,

PT = the whole thing is geod b/c locally geod.

2) "axes are parallel"

$f: t \to d(c(t), c'(t))$



we knew from prev
f is convex

+ periodic

$\Rightarrow$ f is ct.

splitting : $\exists\, u, u_{\mathbb{R}} \in \mathrm{Isom}(Y) \times \mathrm{Isom}(\mathbb{R})$
st $u(y,t) = (u_Y y, u_{\mathbb{R}} t)$

Thm: $\mathbb{Z}^n = \Gamma$ is a grp $\curvearrowright X$ properly by semi-simple isom

elliptic OR hyperbolic.

only prop $\Rightarrow$ can form on $\mathbb{Z}^n \curvearrowright$ even as subgrp.

then X contains Γ-inv subsp isometric to $\mathbb{E}^n$

$\Gamma \curvearrowright \mathbb{E}^n$ by translation,
$\mathbb{E}^n/\Gamma$ cmpt.

Pf (induction)
$n \Rightarrow n+1$
$\Gamma = \langle a_0, \ldots, a_n \rangle$.

a_0 is hyp $\rightsquigarrow$ blc $\curvearrowright$ properso $\rightsquigarrow$ fix'd.

convex $\to$ CAT(0)

$\to$ $Z = \mathrm{Min}(a_0) = Y_0 \times \mathbb{R}$

$\langle a_1, \ldots, a_n \rangle \curvearrowright Y_0$:

hypothesis $\to$ $Y_0 \times \mathbb{R}^n$

Put it together $\rightsquigarrow$ $Z = Y \times \mathbb{R}^{n+1}$.

PT: Min set $\rightsquigarrow$ struct grp.