Def: A grp is CAT(0) $\iff$ it acts geometrically (properly & co-compactly) on a CAT(0) space X.

ex: • $\mathbb{Z}^n \curvearrowright \mathbb{E}^n$) not hyp.

• RAAGs $\curvearrowright$ Univ Cover (Salvetti complex) (CCC)

• uniform lattice in $PSL(2,\mathbb{R}) \curvearrowright \mathbb{H}^2$) not hyp.

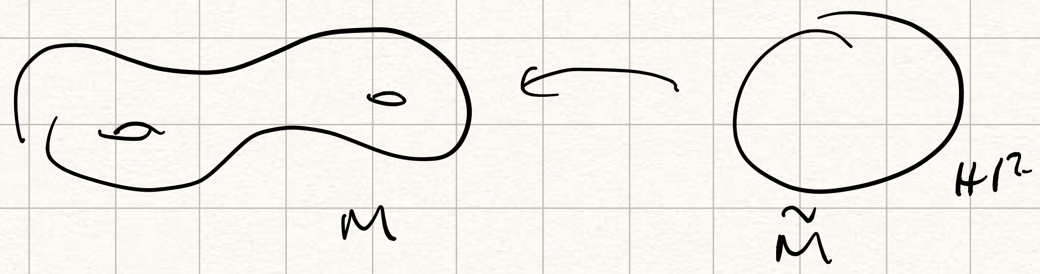
Q (open): Do every hyperbolic grp $\curvearrowright$ geometrically on CAT(0)-space?

not hyp: (• $PSL(2,\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3 \curvearrowright$ Bass-Serre tree

• Uniform lattice $SL(2,\mathbb{R}) \curvearrowright SL(n,\mathbb{R})/SO(n)$

• Free grps $\curvearrowright$ Cayley graph

• M Riemann. manifold w/ $K \leq 0$, $\Gamma = \pi_1(M) \curvearrowright X = \tilde{M}$.



- Uniform lattices in $SL(n, \mathbb{Q}_p)$
  $\circlearrowright$ Euclidean building

<u>non-ex</u> : • $BS(1, n)$
- infinite direct product of finite grps.
- Infinite Higman grps (have $BS(1,n)$ as subgrps
- free Burnside grps.

<u>Note</u> : • $B(1,2)$ not CAT(0) b/c intuitively $|a^n| \approx \ln n$
for the same reason, nilpotent grps are not CAT(0).

formally, $\tau(a) = \tau(a^2) = 2\tau(a)$
$\hookrightarrow$ not semi-simple

Decision problems

(Dehn.)

Thm: CAT(0) grps have solvable word problem in quadratic time.
(NOT known if it's $\Leftarrow$)

X is CAT(0) space $\circlearrowleft \Gamma$ geometrically.

Claim: $\exists C > 0$, finite set $S \subset \Gamma$,
$R \subset F(S)$ st

(1) S generates Γ
(2) every elmt in R is trivial in Γ
and w/ len ≤ 4,
(3) $\forall w \in F(S)$, $w = 1$ in Γ,

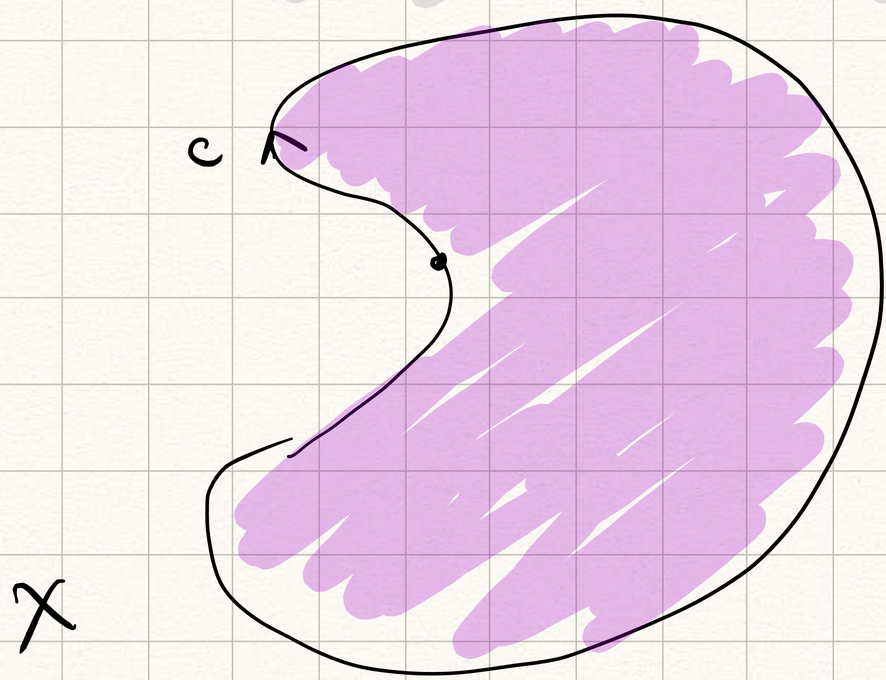
$\Leftrightarrow$ w has form $\quad w = \prod_{i=1}^{N} u_i r_i u_i^{-1}$

$r_i \in R$
$|u_i| \leq C|w|$
$N \leq C|w|^2$

PT: the len of product, and len of conjugator is predetermined by $w \rightsquigarrow$ solvable WP.

$\Rightarrow \Gamma = \langle S \mid R \rangle$ finitely presented.

Isoperimetric Ineq



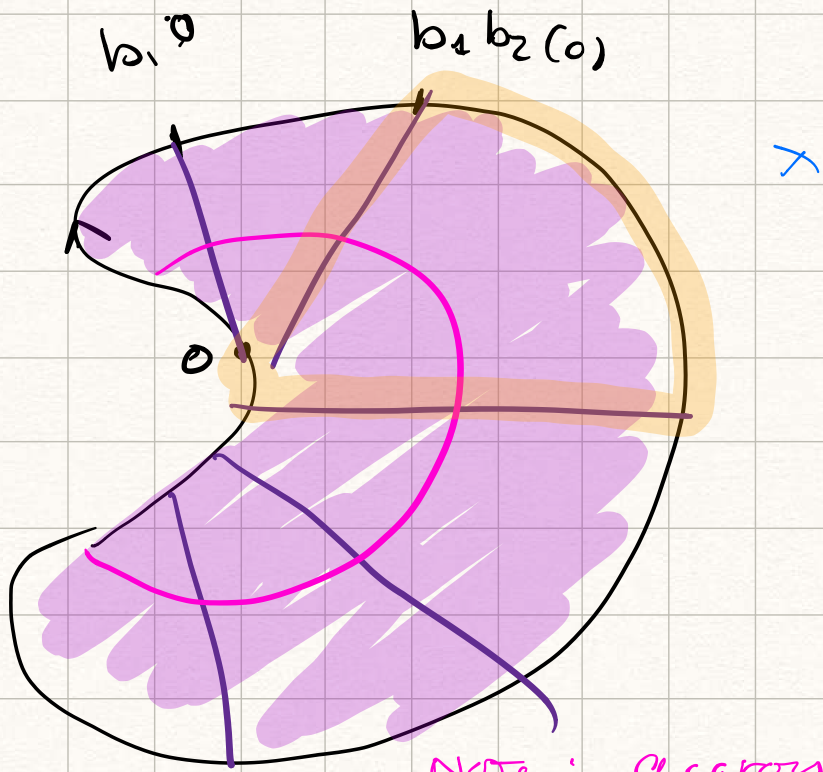
Area(c) = min. area filling curve

in CAT(0) : $\exists C \geq 0$

$$\mathrm{Area}(c) \leq C\,L(c)^2.$$

Now let $w = b_1 \dots b_n$

and let curve be:

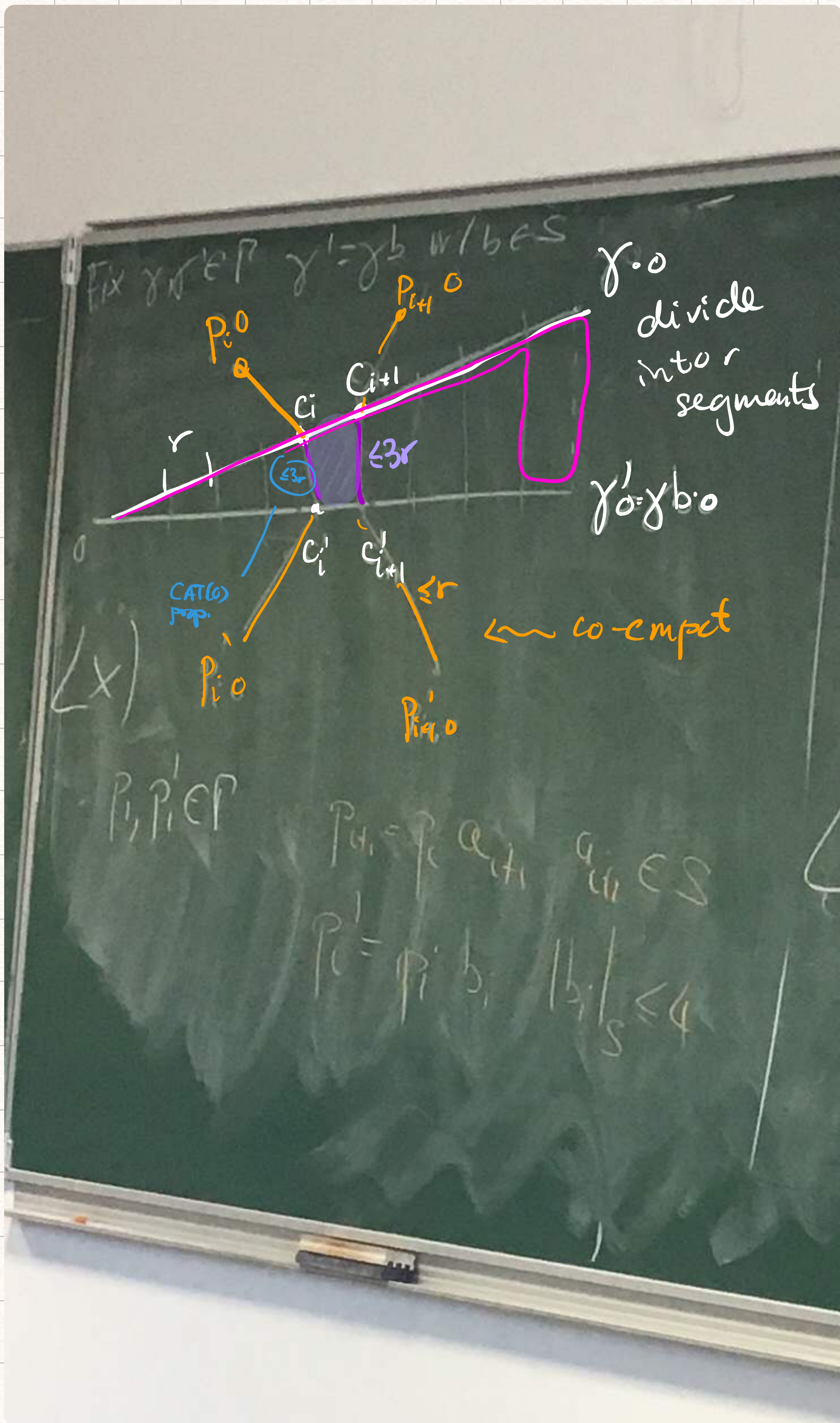


Note: arrows is the relations

Step 0: Find S. fix $o \in X$.

Step 1: $\cdot\ \Gamma \curvearrowright X$ co-cpct,
$\exists\, r \geq 0$ st $\Gamma B(o,r) \supset X$

$\cdot\ \Gamma \curvearrowright X$ proper:
$S = \{\gamma \mid d(\gamma o, o) \leq 3r\}$
is finite. (A)

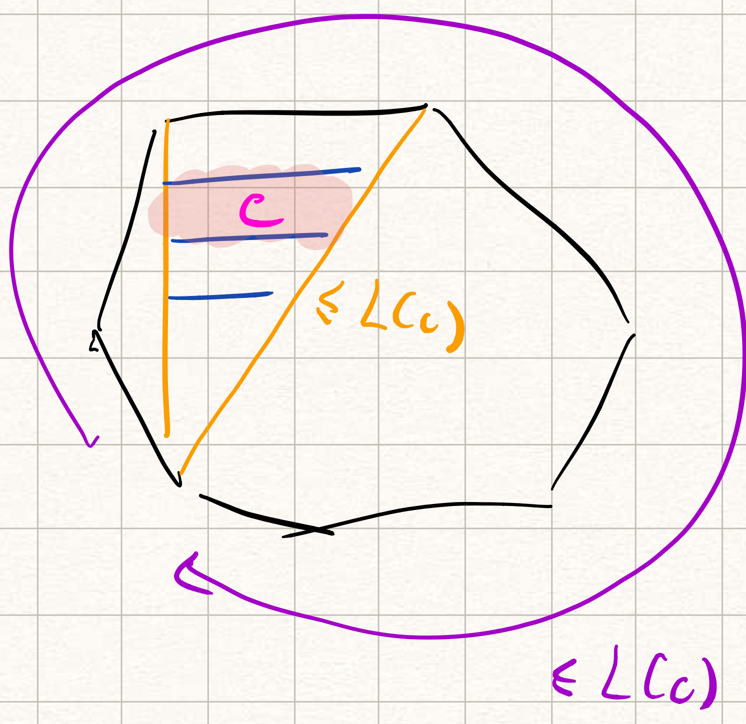

Fix γ, γ' ∈ Γ γ' = γb w/ b ∈ S
γ·o
divide into r segments
P_i o
P_{i+1} o
c_i
c_{i+1}
r
≤3r
γ'o = γb·o
c'_i
c'_{i+1}
≤r
CAT(0) prop.
co-cmpct
P'_i o
P'_{i+1} o
p_i, p'_i ∈ Γ
p_{i+1} = p_i a_{i+1} a_{i+1} ∈ S
p'_i = p_i b_i |b_i|_S ≤ 4

$\gamma b \gamma^{-1} = 1$

$$\gamma^{b} (\gamma')^{-1} = \prod_{i=1}^{\ell} u_i r_i u_i^{-1}$$

$$|u_i|_{S'}\, \ell \le \frac{c}{r} \le d(o, \gamma o)$$

$$\mathrm{Area}(c) \le C L^2(c)$$



Q: Is isom perim $\iff$ Dehn?

A: If G $\circlearrowleft$ geometrically, think yes.