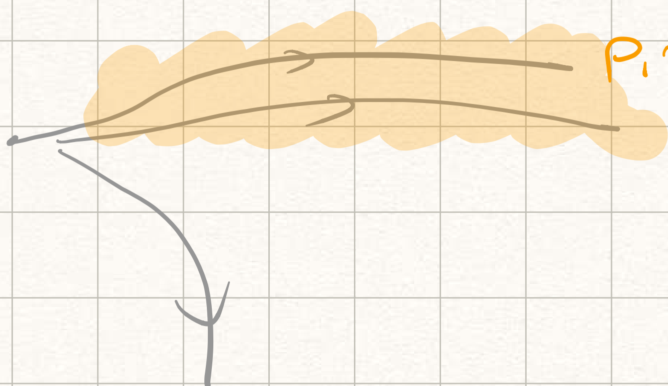
Obs: If X is hyperbolic, geod & proper, take $x_0 \in X$,

$$\partial_{vis} X = \left\{ \begin{array}{c} \rho: [0,\infty) \to X \\ \text{geodesic ray} \end{array} \right\} \Big/ \begin{array}{l} \rho_1 \sim \rho_2 \\ \text{if } \rho_1(t) \text{ remains} \\ \text{at bounded dist of} \\ \rho_2(t) \text{ (w/ cmpt open top).} \end{array}$$



Thm (Gromov classification of isom of hyperbolic spaces)

If X is hyp. and $\alpha \in \mathrm{Isom}(X)$,

- either $\langle \alpha \rangle$ has a bounded orbit (α is elliptic)
- α has a single fixed pt at infinity in this case

$$\|\alpha\| = \lim \frac{1}{n} d(x, \alpha x) = 0$$

(α is parabolic), or

- α has exactly 2 fixed pts at ∞ (in ∂X).

In this last case, $\|\alpha\| > 0$

(α is lox or hyp)

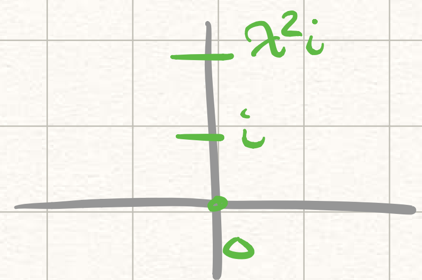
Obs: The limit exists by subadditivity ... does not depend on x (by Δ ineq)

ex: in $\mathbb{H}^2$,

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $-i \mapsto i$ is elliptic;

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $z \mapsto z+1$ is parabolic.

$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$: $z \mapsto \lambda^2 z$

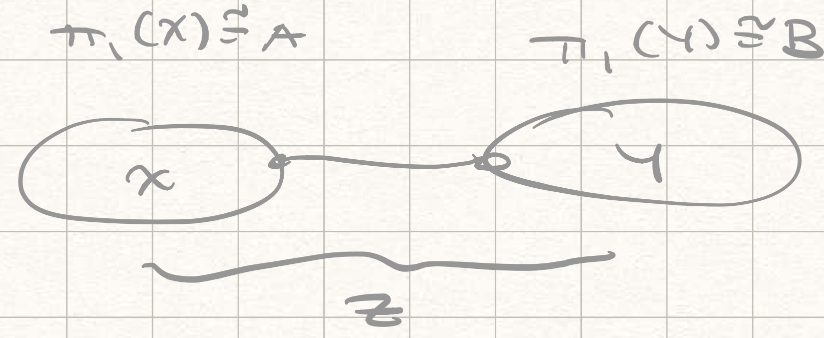


II - Relatively hyp. grps

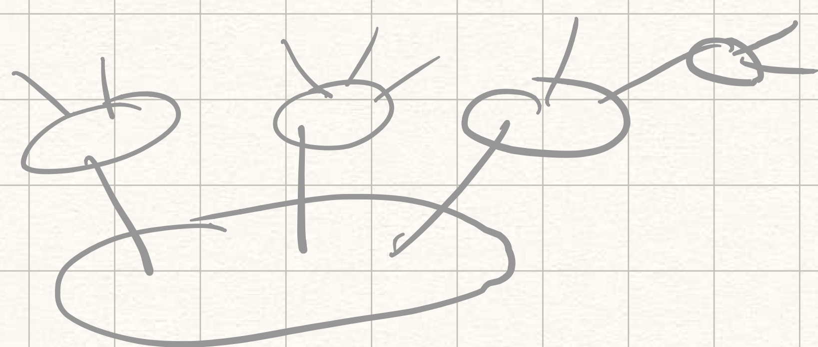
Main ex$^{\underline{o}}$: $A * B$ for A, B arb.

$A * B$ acts on the Serre tree of the free product.

take

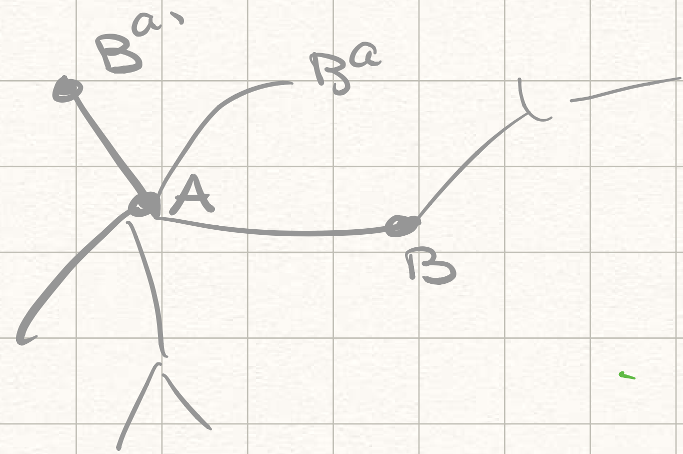


then univ cov $\tilde{Z}$ looks like:



The BS tree is the underlying tree of this tree of spaces, which $\pi_1(Z)$ acts on.

BS tree:

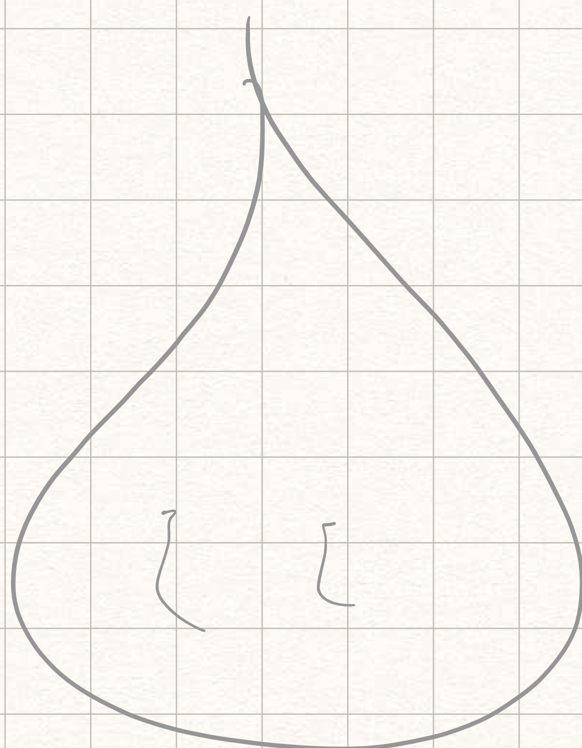


So $\pi_1(Z)$ acts on a 0-hyp space!

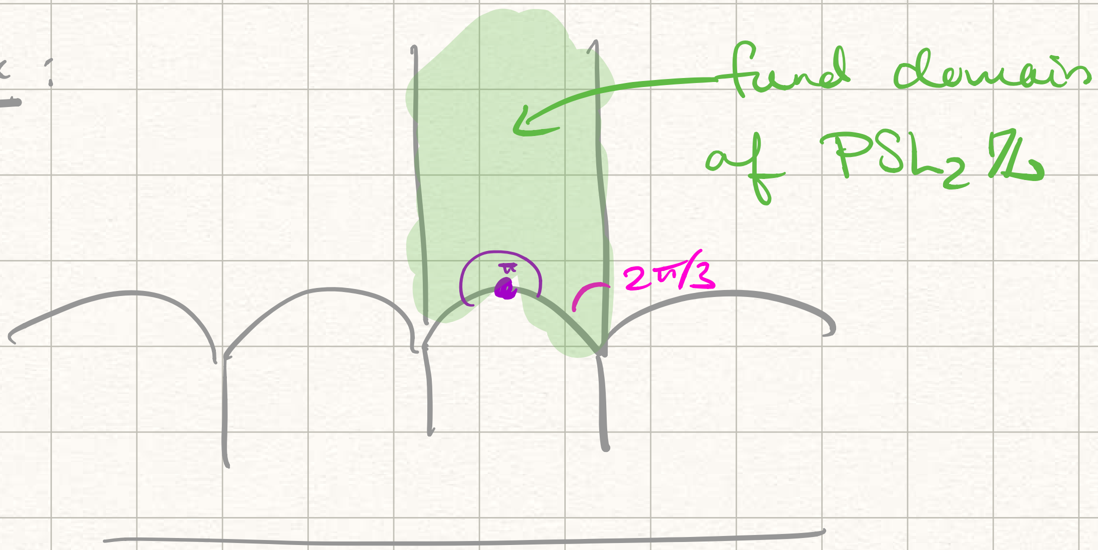
Another main exs:

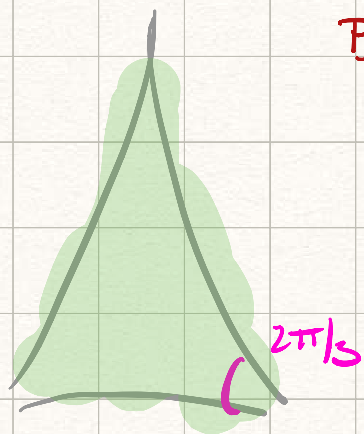
$\Gamma < \mathrm{Isom}\,\mathbb{H}^n$, discrete w/ cofinite vol.

$\Gamma \backslash \mathbb{H}^n$



ex :

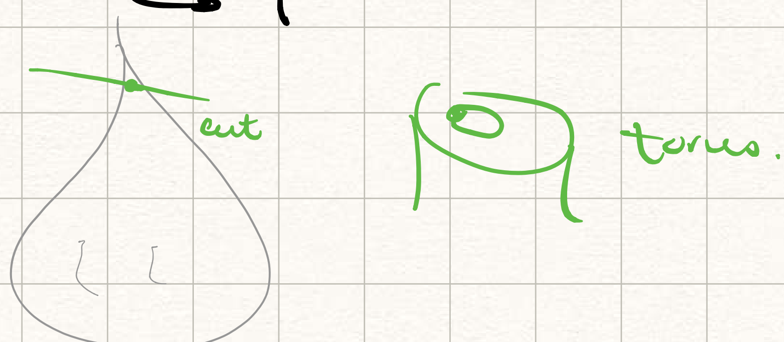




Pf : this has finite vol.

- In the cusp of $\Gamma \backslash \mathbb{H}^n$, one can find some incompressible Tori of dimension n,

hence, one can find tori of dim $n-1$
$\Rightarrow \mathbb{Z}^{n-1} \hookrightarrow \Gamma$



exc: If Γ is a hyperbolic grp, then $\mathbb{Z}^2 \not\hookrightarrow \Gamma$.

Def (Bowditch)

G is **hyperbolic relative** to a subgrp $P < G$ if:

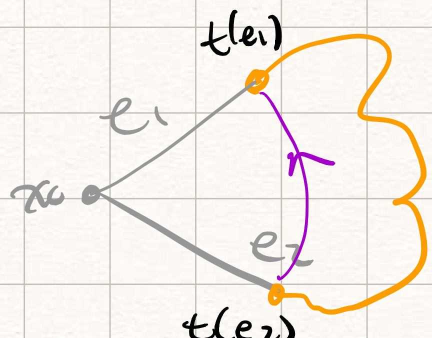
G acts on a hyp. graph w/:

- fin. many orbits of edges
- fin. edge stabs.
- vertex stabilizers infinite or conj to P

& - angularly loc. finite ("fine")

ex: if e_1, e_2 are two edges w/ origin $o(e_1) = o(e_2) = x_0$

$\mathrm{Ang}(e_1, e_2)$ = inf of lengths of path from $t(e_1)$ to $t(e_2)$ not going through x_0.



Ang. loc. fin: $= \forall e$ edges, $\forall \theta > 0$,

$$\left\{ e' \;\middle|\; \begin{array}{l} o(e') = o(e) \\ \mathrm{Ang}(e', e) < \theta \end{array} \right\} \text{ is finite.}$$

for all edge w/ same start, only have fin. many close. $\forall$ nbhd θ.

Problem w/ def: graph may not be loc. fin.

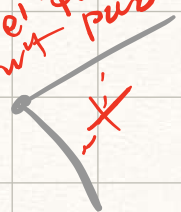
ex: The tree T for the free prod $A * B$.

If A is infinite

$\Rightarrow$ Not locally finite

(Too many edges

No path from $e \to e'$ that doesn't pass by $o(e)$.



$\Rightarrow \mathrm{Ang}(e, e') = +\infty$

b/c inf of empty set.

$$\Rightarrow \left\{ e' \;\middle|\; \begin{array}{l} o(e') = o(e) \\ \mathrm{Ang}(e', e) < \theta \end{array} \right\} \text{ is indeed finite,}$$

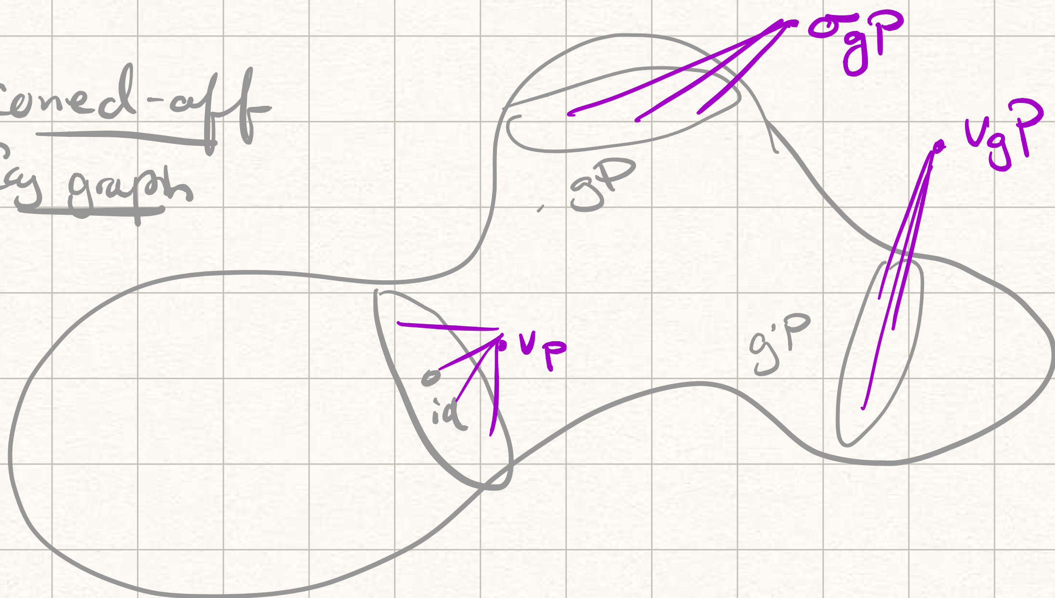
and contains only e.

$\Rightarrow$ So T is ang. loc. fin (even though not loc. fin) $\square$.

Note: NOT an alt def just an obs.

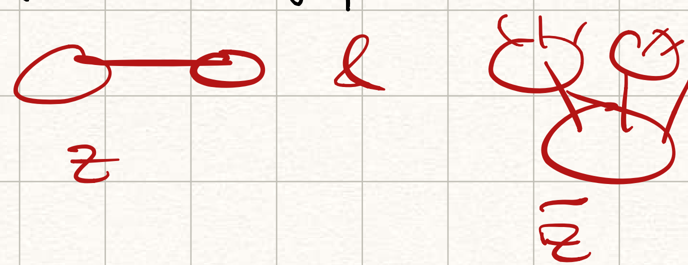
Obs: If G is hyp. rel. to P, ~~then~~ X can be chosen to be the coned-off Cay graph over the left coset of P.

coned-off Cay graph



This coned-off graph is hyperbolic.
and "fine" $\iff \exists$ graph X w/ this property.

Note:



Z w/ collapsed coset roughly is q.i. to Z.

Def 2: G is hyp. rel to $P \subset G$ if there exists X a proper hyp. len. space w/

$G \curvearrowright X$ prop. disc. by isoms

and $\mathcal{H}$ an invariant sys of separated horoballs st

- $G \curvearrowright (X \setminus \mathcal{H})$ co-cmpct — remove the horo & act co-cmpctly.
- Stab of horo are conj to P.

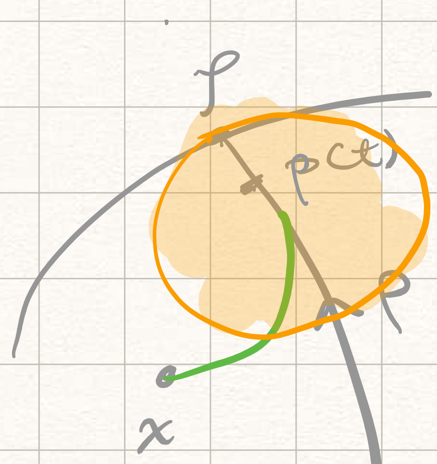
Def: What's an horoball: → take limit pt, take geod to pt.

take $\xi \in \partial X$. Take $\rho : [0,\infty) \to X$ geodesic ray to ξ.

$h : X \to \mathbb{R}$,

$x \mapsto \lim_{t \to \infty} d(\rho(t), x) - t$

Busemann fn



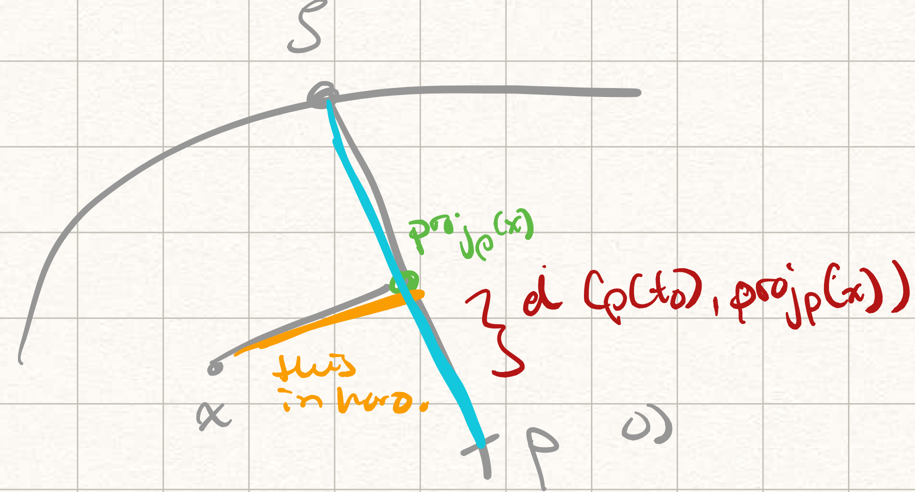
A **horoball** is B st $\exists D_1, D_2$

$h(x) \le -D_1$ in B

$\ge D_2$ outside

or otw → pt = count pts that are deep in p

PT: $x \in B$ if $d(x, p) \le d(p(t_0), \overbrace{proj_p(x)}^{\text{proj of } x \text{ on } p.})$



ex: in H^2, horoballs at the pt $\infty \in \mathbb{R} \cup \{\infty\}$ are slightly upper half-spaces.

Thm (Bowditch) The two defs of rel. hyp are equiv.

General Rmk: The situation is good for "parmanence prop":

if P satisfies "pty $\hat{\boxtimes}$"
and all hyp. grps satisfying pt $\hat{\boxtimes}$, then one expects that, if G is hyp. rel. to P, G satisfies $\hat{\boxtimes}$.

ex: P hyp, & P rel. hyp. G, then G hyp.