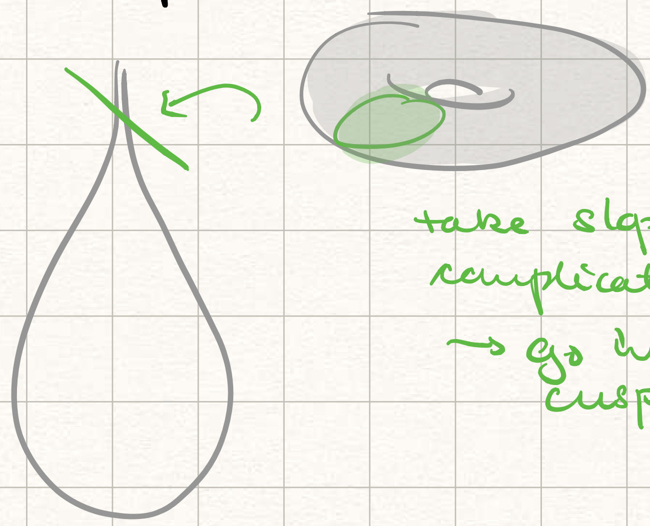
What happened:



take slope v. complicated → go high in cusp!

The more complicated the Dehn filling, the less you change in some sense.

Pf: Change the metric very little.

VI - Acyl. hyperbolicity

Def: Let G be a grp acting on a δ-hyp. space X. The action is acylindrical if $\exists R>0$ st $\forall x,y$, $d(x,y)>R$,

$$\left\{ g \;\middle|\; \begin{array}{l} d(gx,x)\le 100\delta \\ d(gy,y)\le 100\delta \end{array} \right\} \text{ is finite.}$$

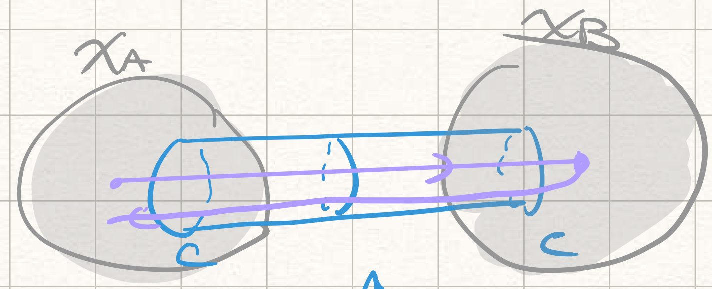
ex: • proper actions

($\forall x$, $\{g \mid d(gx,x) \le 100\}$ is finite})

• $A \underset{\mathbb{Z}}{*} B$ st $\forall a \in A \setminus C$,

(*) $aCa^{-1} \cap C = \{1\}$

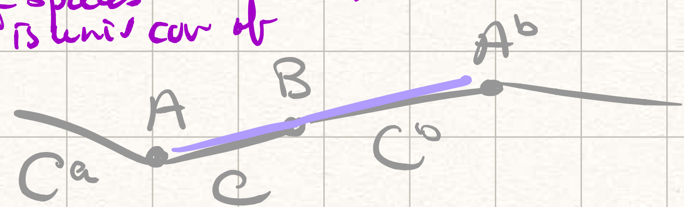
(w/ $C \cong \mathbb{Z}$)



$\mathbb{Z}$-embedding.

BS-tree:

(tree of spaces that is univ cov of

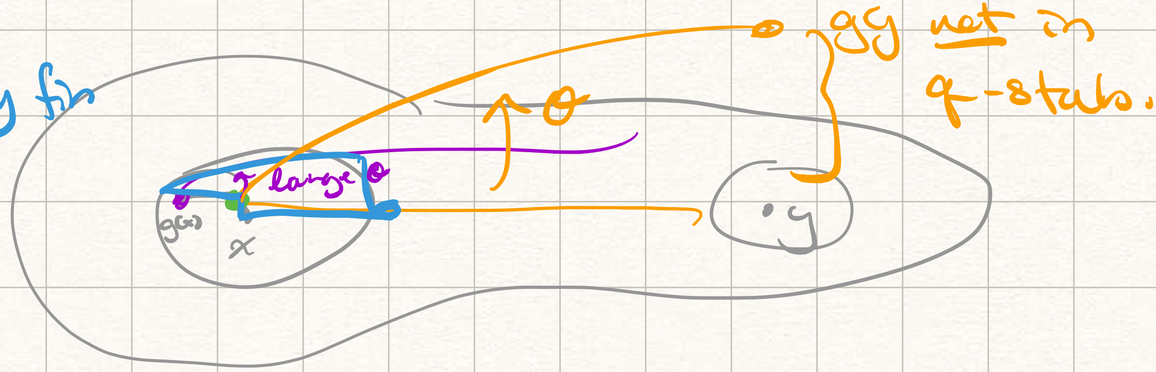


(*) $\Rightarrow$ no elmt fixes two edges.

$\Rightarrow$ "acylindrical":
b/c when I try to embed a cylinder, stop after two steps.

ex: Relatively hyp. grps acting on their hyperbolic "fine" graph.

non-locally fin graph, ∞-many elmt in q-stab(x).



Need to check that elmts in q-stab(x) $\cap$ q-stab(y) = finite.

Take g: $d(g(x), x)$ - small, goes through ∞-valence • & make large angle.

Now send $y \mapsto g(y)$. $\Rightarrow$ will make big angle $\Rightarrow$ gy far from y.

? didn't really get it 2nd time.

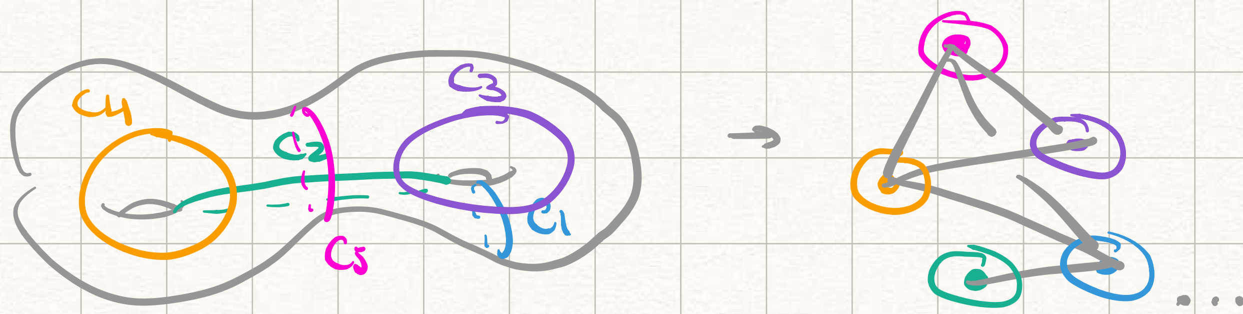
ex: $MCG(\Sigma_g)$, $g \geq 2$. where

$$MCG(\Sigma_g) := \mathrm{Diffeo}^+(\Sigma_g) / \mathrm{Diffeo}^0(\Sigma_g)$$

Dehn-Nielsen:

$$MCG(\Sigma_g) \simeq \mathrm{Out}(\pi_1(\Sigma_g)) = \mathrm{Aut}(\pi_1(\Sigma_g, x_g)) / \mathrm{Inn}(\pi_1(\Sigma_g))$$

Def: The **curve graph** of Σ_g is the graph whose vertices are homotopy classes of s.c.c. and edges are $([c_1],[c_2])$ where c_1 and c_2 can be realized as disj.



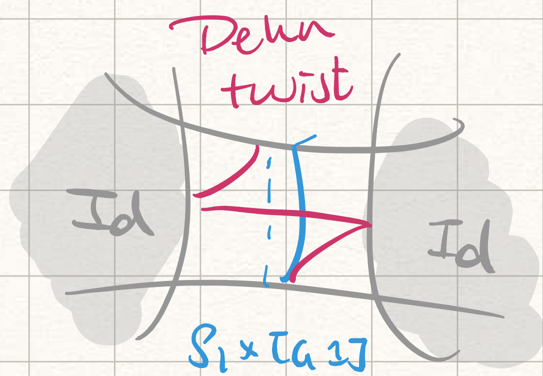
...

Facts: The curve graph of $\Sigma_g, g \geq 2$ is infinite, connected, and of infinite diameter.

Thm: The curve graph $\Sigma_g, g \geq 2$ is Gromov hyp.
(Bowditch, Maz-Minsk) The action of $MCG(\Sigma_g)$ on it is acyl.

Obs: $MCG(\Sigma_g)$ are NOT rel. hyp (except to themselves).

Indeed, $MCG(\Sigma_g)$ is generated by Dehn twist along some system of curves.



→ Dehn twist on disjoint curves commute due to disj support.

⟹ Any two disj. curves produce $\mathbb{Z}^2 \hookrightarrow MCG(\Sigma_g)$.

? ~ still don't get it.

Let $g \in \mathbb{Z}^2$. Then g has ∞-order
→ parabolic (no bounded orbits)
↳ but then the whole $\mathbb{Z}^2$ has to be parabolic dw. (1-fix pt on ∂G)

If $MCG(\Sigma_g)$ was rel. hyp, this $\mathbb{Z}^2$ would be in parabolic.

$\mathrm{Parab}^{C(G)}$ = other grps which intersect at ∞-pts.

Changing curves one at a time shows that MCG is contained in a single parabolic.

$\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$

they all intersect $\Rightarrow$ same parabolic.

$\Rightarrow MCG(\Sigma_g)$ is NOT rel. hyp $\square$.

Thm (Thurston)

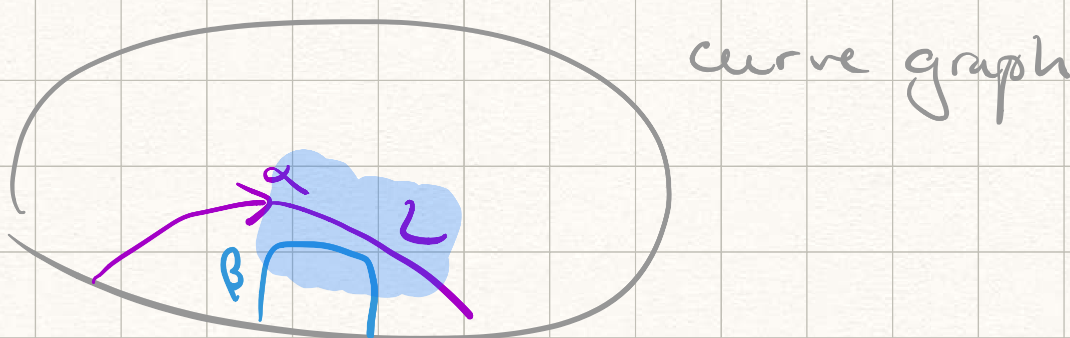
If $\alpha \in MCG(\Sigma_g)$ either:

i) Some power $\alpha^m, m>0$ preserves a S.C.C.

ii) α is pseudo-Anosov & it is lox. isometry on the graph of curves.

Obs (from acyl):

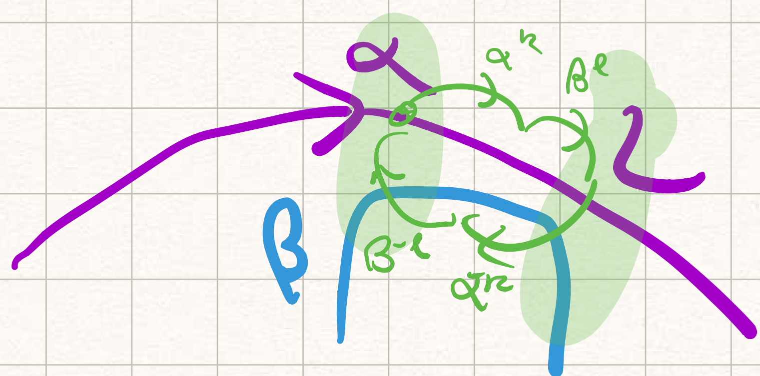
If $\alpha \in MCG(\Sigma_g)$ pseudo-Anosov, let L be a quasi-axis for α,



let $\beta = \phi \alpha \phi^{-1}$ st $\phi(L) \neq L$.

(Q: Can α, β be parallel for long time?
A: No)

then have that $\underset{10\delta}{N(L)} \cap \underset{20\delta}{N(\phi L)}$
will be unif bounded (wrt ϕ).



if parallel for long time:
$[\alpha^k, \beta^l]$ quasi-fixes two pts
far away in this intersection

$\to$ contradicts axyl.

Applications

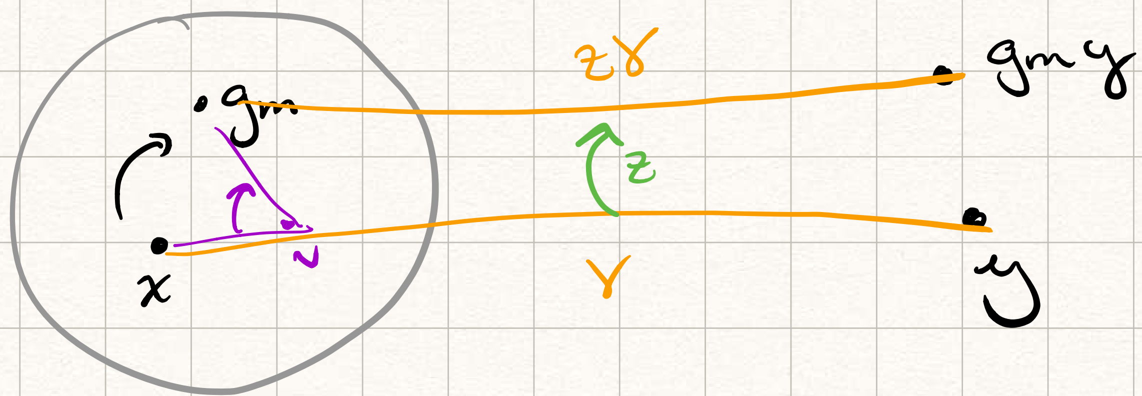
Thm (Dahm. + Guirardel + Osin)

"A Dehn filling thm for some subgrp of $MCG(\Sigma_g)$"

$MCG(\Sigma_g)$ is SQ-universal, ie, any countable grp is a subgrp of a quotient of $MCG(\Sigma_g)$.

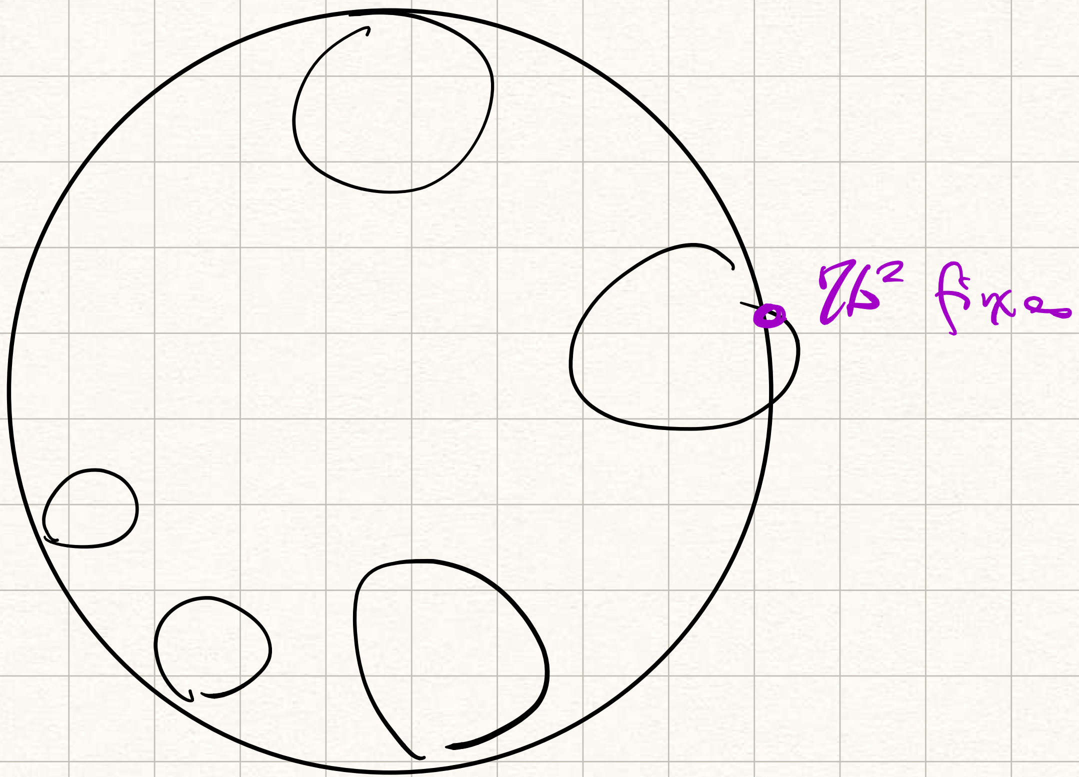
(Done by Dehn filling free subgrp gen by pseudo-Anosov).

Ans: Suppose you have an infinite q-stab. The graph is NOT locally finite $\Rightarrow$ must go through ∞-valence pt w/ large angle



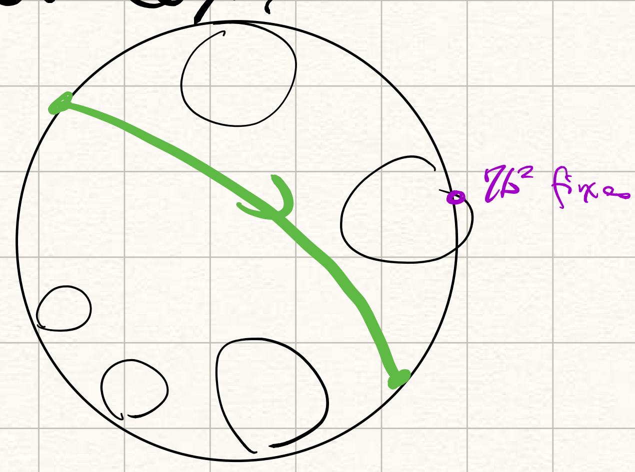
Let γ be geod b/w x & y, & suppose $d(g_n y, y)$ is small.

Suppose that $z\gamma$ makes a small angle. Then



1) $\mathbb{Z}^2$ being parabolic
means it's a conj of P
(b/c fix + boundary pt horo,
and unbounded orbit close to
fixed pt).

2) $\mathbb{Z}^2$ <u>is</u> parabolic b/c if
it were to be lox, then
it would act lox,
which means
that the
centralizer
is close
to it



$\Rightarrow$ contra to properness on horoball.