Computing in fg groups

Resources: · C. Sims; Computation w/ FP groups (formal + pseudo)
· D. Holt et al; Handbook of Computational grp th.

Online - A. Hulkes - Notes on Compu. grp theo.
- D. Holt - lecture notes on pres. grp.

- M. Neunhöffer ~ 2013 lectures

Grp rep on CPU

- (finite grp) permutations
- matrices over field
- power-conjugate pres.
- finite pres & generators & defining relators.

NO canonical rep. so use 3 prev to figure it out

Notation

$G = \langle X \mid R \rangle$, $A = X \cup X^{-1}$ (alphabet), $G := F(X) / \langle\langle R \rangle\rangle$.

relation: $w_1 = w_2$, relator: $w_1 w_2^{-1}$ $(=1)$

Tietze transformation: replace gen & relations.

$G = \langle a, b \mid aba = bab \rangle$

$G \cong \langle x, y \mid x^3 = y^2 \rangle$

thm: If $G \cong H$, then $G \xleftrightarrow{\text{Tietze}} H$

(very simple proof b/c knowing this doesn't tell you how to do it)

the problem.

Dehn Problem:

- If answer is yes, can prove so using Tietze, but if No, then computer runs forever.
- Was interested in surface grps.

Problem: Computing the largest abelian quotient.

$$G_{ab} = G / [G,G]$$

$$G_{ab} := \langle X \mid R \cup C \rangle, \quad C = \{ [x,y] : x,y \in X \}$$

→ Switch to additive notation.

elements = row vector.

ex: if relation is: $a(bd^{-1})^2$

$$\downarrow$$

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ & \cdot & & \cdot\ \cdot \end{pmatrix}$$

& row-reduce. ⇝ Smith Normal Form.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ & & \cdot\ \cdot & \end{pmatrix}$$

$$G_{ab} = \mathbb{Z}/\mathbb{Z} \oplus \mathbb{Z}/(2\mathbb{Z}) \oplus \cdots$$

Problem: if matrix not sparse, the size of problem grows exponentially

Solution: LLL algorithm.

Others:
- finite p-grps
- nilpotent
- polycyclic
- solvable quotients

$\Bigg\}$ ~ linear alg operation!

look into: basic computer algebra

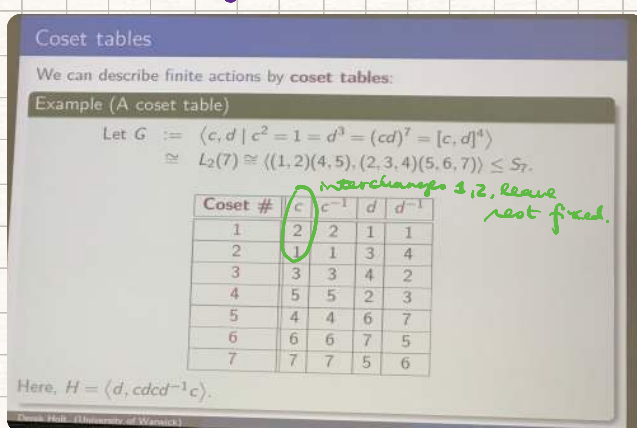
Subgroups of finite index (finite quo)

subgroup fi $\longleftrightarrow$ transitive action.

$$\{H \leq G : [G:H] < \infty\} \xrightarrow{\simeq} \{G \xrightarrow{\text{ie}} S_n \; -- \quad \}$$

Coset tables

Can describe finite actions by coset tables



Coset tables

We can describe finite actions by **coset tables**:

Example (A coset table)

Let $G := \langle c, d \mid c^2 = 1 = d^3 = (cd)^7 = [c,d]^4 \rangle$
$\cong L_2(7) \cong \langle (1,2)(4,5), (2,3,4)(5,6,7) \rangle \leq S_7$.

interchanges 1,2, leave rest fixed.

Coset #	c	c^{-1}	d	d^{-1}
1	2	2	1	1
2	1	1	3	4
3	3	3	4	2
4	5	5	2	3
5	4	4	6	7
6	6	6	7	5
7	7	7	5	6

Here, $H = \langle d, cdcd^{-1}c \rangle$.

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Main : Todd - Coxeter coset enum

trivia : used to be done by hand!

Let $G = \langle X \mid R \rangle$

$H = \langle h_1 .. h_k \rangle < G$

Don't know if index is finite

Idea : Construct permutation action of G on right coset of H.

- Assign number to cosets, & make sure that
  - → mult by elements of R fix all cosets
  - → mult by H fix first coset (first coset is H)

If index is finite will halt.

Coset enumeration - a worked example

Let $G := \langle a, b \mid a^3, b^3, abab \rangle$ and $H := \langle a \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	1	1	2	3
2	Hb	3	4	3	1
3	Hbb	4	2	1	2
4	Hba^{-1}	2	3	4	4

H puts b in #2

We start with an empty table like this.

Events:

- $1a = 1$ from $a \in H$ $(Ha = H)$
- $Hb \rightarrow$
- $3 = 2b = Hbb$
- $3b = 1$ b/c, $3b = Hb^3 = H = 1$
- $3 = 2a$ from $1abab = 1$
- $4 := 2a^{-1} = 3a$ from $2a^3 = 2$

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Once the table is closed → constructed perm.

Thm: If index finite, the table eventually closes.

Found perm-rep on 4 pts, w/ H fixing 1st.

Def: coincidences

sometimes, get $k = ia$, for $k, i \in \mathbb{N}$, $a \in A$

→ same coset.

Have to (if $k > j$) replace every k w/ j.

sometimes new coincidences happen as result.

* Tricky to program efficiently!
number of "technical pitfalls"
(? edge-cases?)

Def: strategy

what to define next.

time vs space complexity

1. HLT
   define cosets while scanning relators
   fast but lots of coincidences.
2. Felsch
   filling rows & make all possible deduct

(avoid coincidences, slower)

③ ACE

allow to combine: $\underline{ex}$ 60 defs HLT,

40 Felsch.

No overall optimal strategy.

$\underline{Q}$: can we find out using random grp? Ans: most of them are hyp. which may not be what we care about.

$\underline{\text{low-index subgrp}}$: conj. class of subgrps of index up to n.