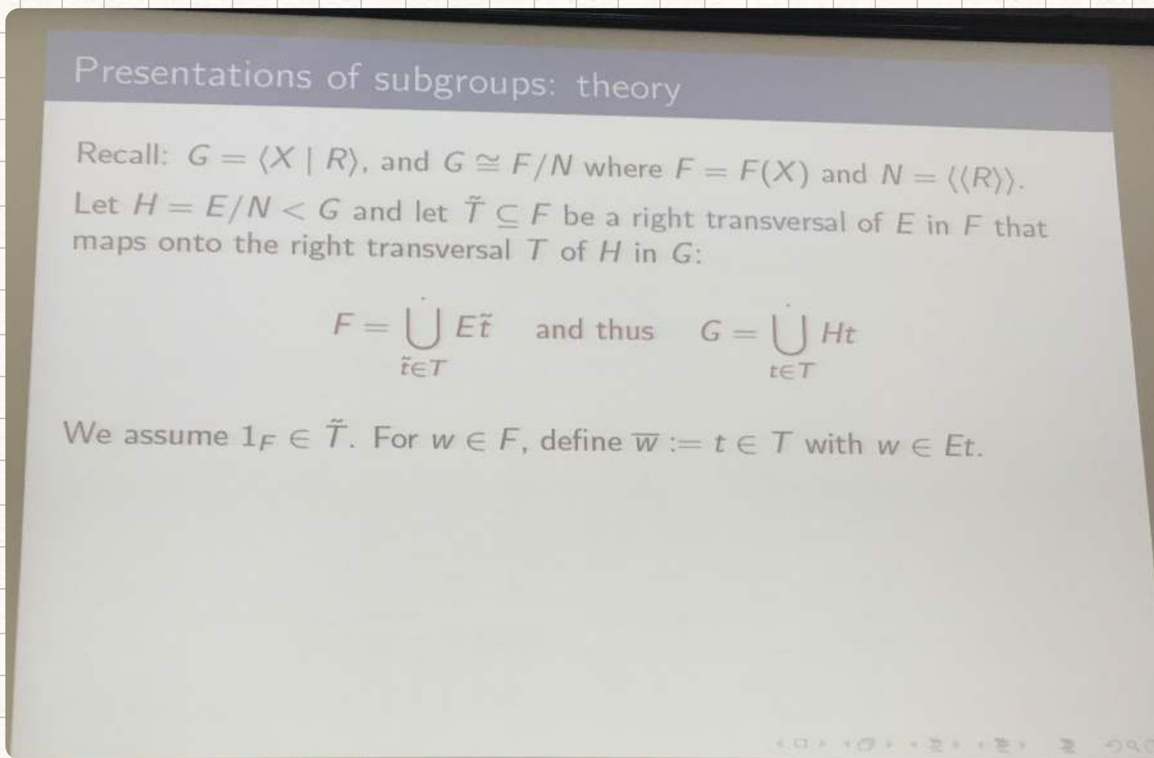
- Coset enum works if $|G:H|$ finite,

  but this is unsolvable (so procedures $\to$ not alg).
- Can do: · compute all subgrps of index $\leq n$. (fixed)

  · finding pres if order & mult table known.



Thm (Nielsen – Schreier)

$E = \langle Y \rangle$ and hence $H = \langle yN : y \in Y \rangle$, where

✓ coset rep

$$Y := \{ tx(\overline{tx})^{-1} \mid t \in T, x \in X, tx \neq \overline{tx} \} \subseteq F \setminus \{1_F\}.$$

If T is prefix-closed, then E is freely gen by Y.

Thm (Reide – Schreier)

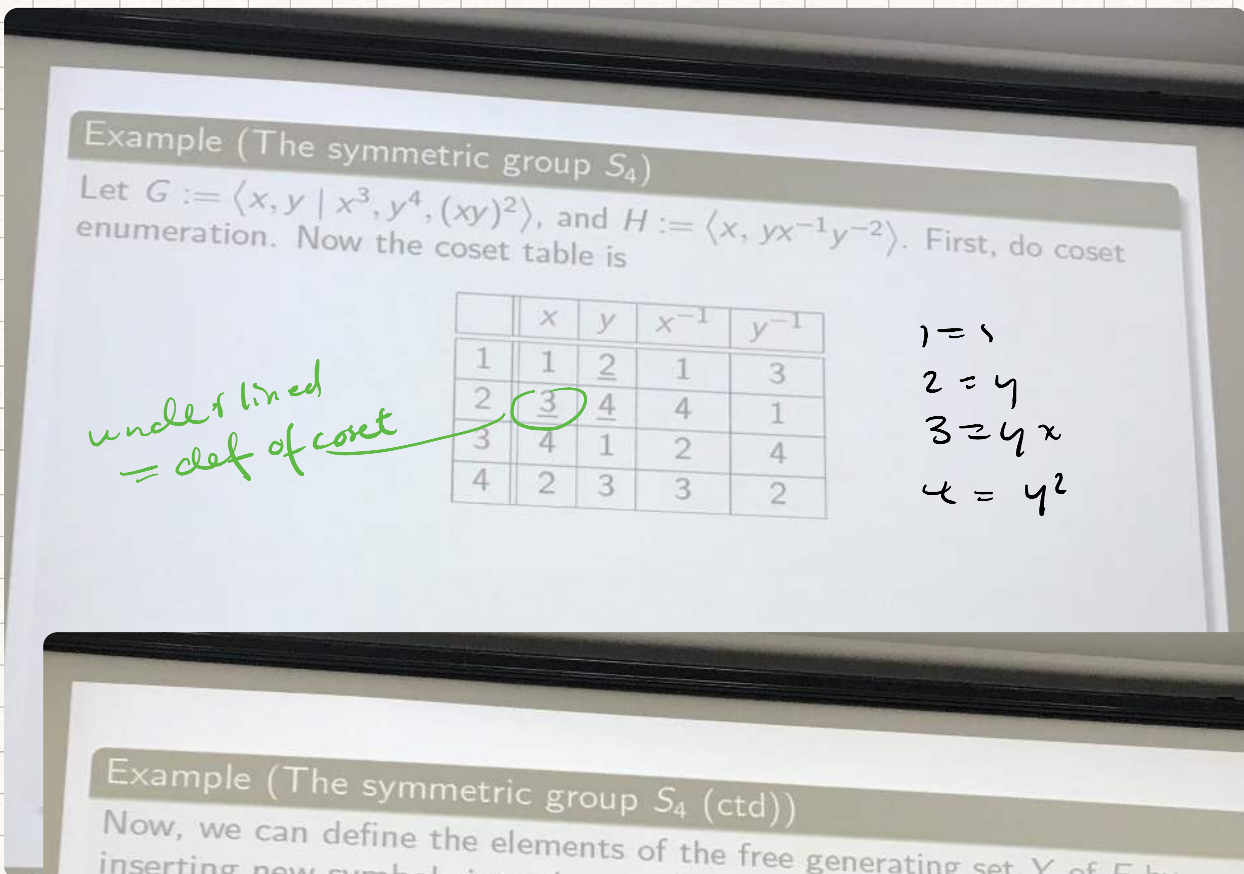
Let G, F, H, E, T above & prefix closed.

Then

$$H \cong \langle Y \mid \rho(t w t^{-1}) \ \forall\, t \in T,\ w \in R \rangle$$

$\rho : E \to F(Y)$ is isom mentioned above.

Proof (ex)



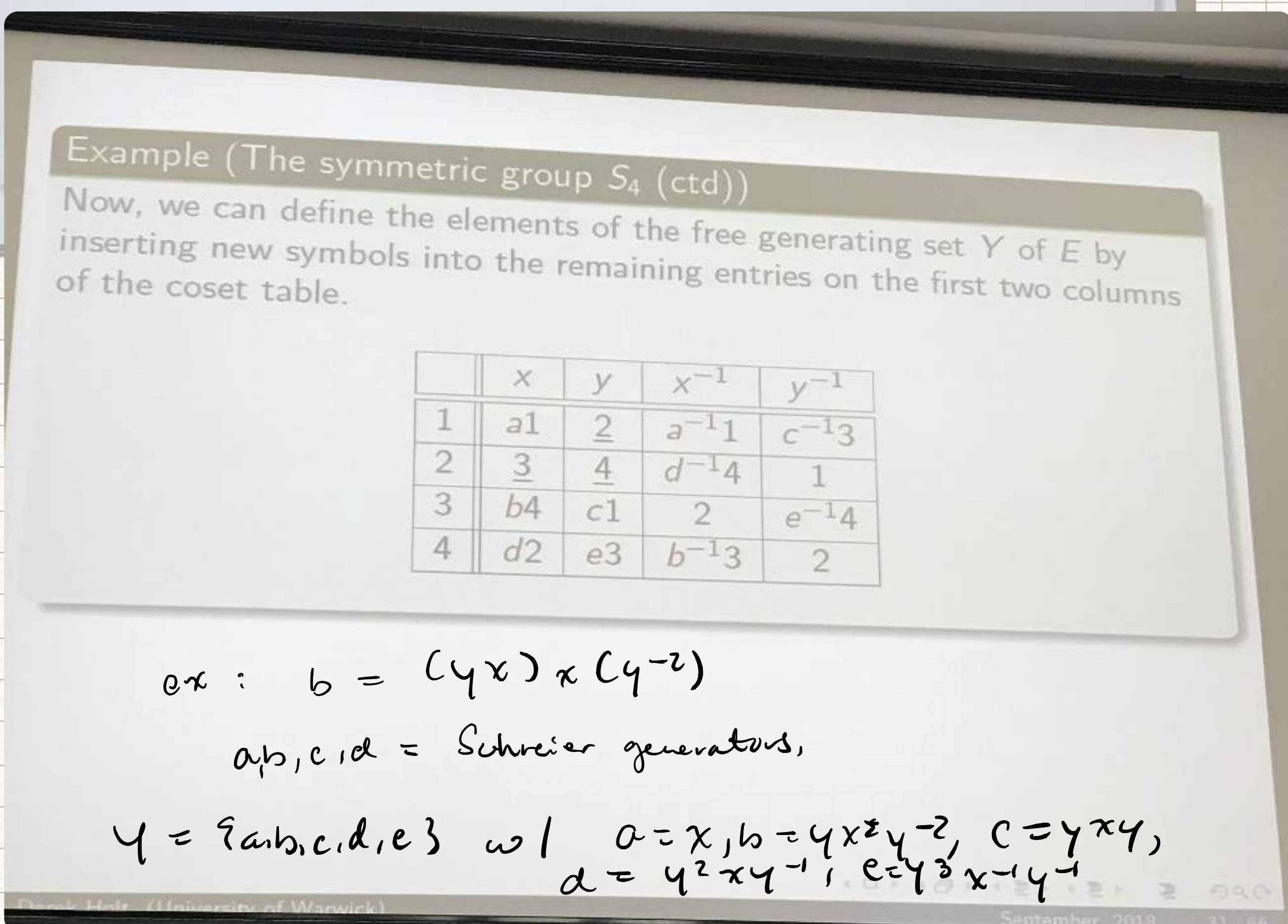
Example (The symmetric group S_4)

Let $G := \langle x, y \mid x^3, y^4, (xy)^2 \rangle$, and $H := \langle x, yx^{-1}y^{-2} \rangle$. First, do coset enumeration. Now the coset table is

	x	y	x^{-1}	y^{-1}
1	1	<u>2</u>	1	3
2	<u>3</u>	<u>4</u>	4	1
3	4	1	2	4
4	2	3	3	2

underlined = def of coset

$1 = 1$
$2 = y$
$3 = yx$
$4 = y^2$



Example (The symmetric group S_4 (ctd))

Now, we can define the elements of the free generating set Y of E by inserting new symbols into the remaining entries on the first two columns of the coset table.

	x	y	x^{-1}	y^{-1}
1	$a1$	<u>2</u>	$a^{-1}1$	$c^{-1}3$
2	<u>3</u>	<u>4</u>	$d^{-1}4$	1
3	$b4$	$c1$	2	$e^{-1}4$
4	$d2$	$e3$	$b^{-1}3$	2

ex : $b = (yx)\,x\,(y^{-2})$

a, b, c, d = Schreier generators,

$Y = \{a, b, c, d, e\}$ w/ $a = x,\ b = yx^2y^{-2},\ c = yxy,$
$d = y^2xy^{-1},\ e = y^3x^{-1}y^{-1}$

Now compute $\rho(twt^{-1})$'s.

Example (The symmetric group S_4 (ctd))

We calculate the relators $\rho(twt^{-1})$ of H by scanning the cosets of H in G under the relators of G using the modified coset table.

	x		x		x	
1		$a1$		$a^2 1$		$a^3 1$
2		3		$b4$		$bd2$

$yx3 = bdy$

$yx^3y^{-1} = bd$.

	y		y		y		y	
1		2		4		$e3$		$ec1$

	x		y		x		y	
1		$a1$		$a2$		$a3$		$ac1$
3		$b4$		$be5$		$beb5$		$(be)^2 2$
4		$d2$		$d4$		$d^2 2$		$d^2 4$

Now get presentation..

Example (The symmetric group S_4 (ctd))

So

$$H \cong \langle a, b, c, d, e \mid a^3, bd, ec, ac, (be)^2, d^2 \rangle$$

We can now perform Tietze transformations on this presentation and use the relators bd, ec, ac to eliminate $d = b^{-1}$, $e = c^{-1}$, $c = a^{-1}$, to give the presentation

$$H \cong \langle a, b \mid a^3, (ba)^2, b^2, \rangle$$

which is a standard presentation of the dihedral group of order 6 (and also of the symmetric group S_3).

Q : Can you do Tietze while running alg?

A : Yes. Ppl have tried.

- Can also find pres w/ <u>user-supplied</u> pres, but have to calc subgrp pres. while doing coset enum. If index large, can result in very long relators.

- From D. Holt's exp: user-supplied ~ same as CPU gen.
  - order of operations make big difference.

- Have to deal w/ coincidences (of course).

PROBLEM

Given $G = \langle x \mid R \rangle$.

How to prove $|G| = \infty$?

(for finite, can do coset enum).

SOL: 1) Find abelian invariants → that's easy
(maybe one of them is ∞.

2) • Compute <u>low index subgrps</u>
   • Use Reide-Schreier to find pres.
   • Compute abs. inv for <u>those</u>.
   • If find infinite factor, Then G ∞ also.

Example

$$G := \langle x, y \mid x^3, y^3, (xy)^3 \rangle; \quad H := \langle x^{-1}y, yx^{-1} \rangle.$$

Show that $|G : H| = 3$, calculate a presentation of H and show that H has abelian invariants $[0, 0]$. (In fact $H \cong \mathbb{Z}^2$.)

EVERYTHING SO FAR

Computing invarious quotients of fp grps $G = \langle X \mid R \rangle$.

Next

Solving Dehn in specific grp classes.

I. Small Cancellation Th.

Small canc. $\sim$ two rels don't have overlapping rels.

Dehn: on surface grp w/ small. canc. prop.

Small cancellation theory

Let $G := \langle X \mid R \rangle$, and assume that all $r \in R$ are cyclically reduced. Let $\hat{R}$ be the set of all rotations and inversions of the elements of R.

clos R

For example, if $R = \{a^3, b^3, aba^{-1}b^{-1}\}$, then (with $\bar{a} := a^{-1}$, $\bar{b} = b^{-1}$):

$$\hat{R} = \{a^3, \bar{a}^3, b^3, \bar{b}^3, ab\bar{a}\bar{b}, b\bar{a}\bar{b}a, \bar{a}\bar{b}ab, \bar{b}ab\bar{a}, ba\bar{b}\bar{a}, a\bar{b}\bar{a}b, \bar{b}\bar{a}ba, \bar{a}ba\bar{b}\}.$$

Def : (piece)

non-empty word that is prefix of two dif elements of $\hat{R}$.

ie $pu, pv \in \hat{R}$ $u \neq v$.

Def ($C'(\lambda)$)

Small cancellation theory (ctd)

Example

$$G = \langle a, b, c, d \mid a^7, b^7, c^7, d^7, a^{-1}b^{-1}abc^{-1}d^{-1}cd \rangle .$$

The pieces are $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}$, and G satisfies $C'(1/6)$.

pieces = gens & inv.

every fraction of word piece occupies $< 1/7$.

so $\lambda \geq 1/6$.

Definition (Condition $T(q)$)

We say that $\langle X \mid R \rangle$ is $T(q)$, if the following holds:

- Let $3 \leq h < q$ and $(r_1, r_2, \ldots, r_h) \in \hat{R}^h$ with no successive elements r_i, r_{i+1} or r_h, r_1 an inverse pair. Then at least one of the products $r_1 r_2, r_2 r_3, \ldots, r_h r_1$ is reduced without cancellation.

$T(q)$ says that all vertices in any reduced van Kampen diagram for the presentation have valency at least q.

Thm (Lyndon-Schupp) (use curvature of VK-diag to prove results)

$G = \langle X \mid R \rangle$, $r \in R$ cyclically red,

if: $C'(1/6)$ or

$C'(1/4)$ and $T(4)$ or

$C'(1/3)$ and $T(3)$

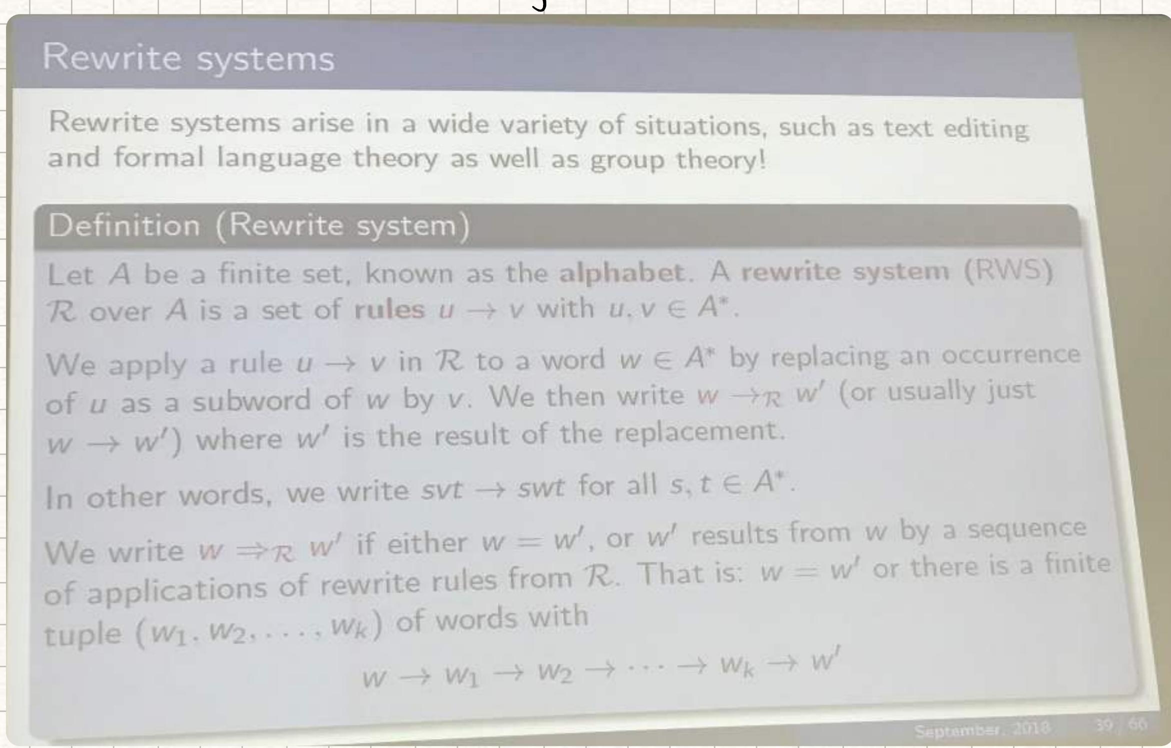
then: Dehn's alg solves WP for $G = \langle X \mid R \rangle$ in linear time.

Keep ex: $G = \langle a, b, c, d \mid a^7, b^7, c^7, d^7, a^{-1}b^{-1}abc^{-1}d^{-1}cd \rangle$

Dehn alg: use Rewrite sys

Rewrite sys (RW)

Sub one str for another.



Note: $w \Rightarrow_R w'$ need NOT be from finite steps of rewrites.

Def (Dehn RW sys)

Given fp, $G = \langle X \mid R \rangle$
write all $r \in \hat{R}$ as $r = ab$ w/
$|a| > |b| \geq |a| - 2$ & define
RW $a \to b^{-1}$ over $A = X \cup X^{-1}$.

Dehn RWS = set of rules w/ $aa^{-1} \to 1$
$\forall a \in A$.

ex: $abcde \in R$
$\downarrow$
$abc = e^{-1}d^{-1}$
$\Rightarrow abc \to e^{-1}d^{-1}$

Dehn's alg solves WP
if $\forall \underset{A^*}{\overset{\Uparrow}{w}}$: $w =_G 1 \iff w \Rightarrow_R 1$.

irrespective of the order of rules.

"confluent on id"

· RW rules length red $\to$ solve WP in linear time.
· Works for when Grp satisfies some small cancellation property.

Dehn's algorithm: example

! (Small error somewhere)

Consider the word $w = a^3b^6abcd^{-1}cd^{-6}a^4$ in the example above. Then

$$a^{-1}b^{-1}abc^{-1}d^{-1}cd \in R \Rightarrow abc^{-1}d^{-1}cda^{-1}b^{-1} \in \hat{R}$$

so there is a rewrite rule $abc^{-1}d^{-1}c \to ba^{-1}d^{-1}$.

Applying this to w gives

$$w = a^3b^6\underline{abcd^{-1}c}d^{-6}a^4 \to a^3b^6\underline{ba^{-1}d^{-1}}d^{-6}a^4 = a^3b^7d^{-7}a^4.$$

Now $b^7, d^{-7}, a^7 \in \hat{R}$ yield rewrite rules $b^4 \to b^{-3}$, $d^{-4} \to d^3$, $a^4 \to a^{-3}$, and applying these to w, followed by applications of $b^{-1}b \to 1$, etc, gives:

$$a^3b^7d^{-7}a^4 \to a^3b^{-3}b^3d^3d^{-3}a^4 \to a^7 \to a^{-3}a^3 \to 1$$

so $w =_G 1$.

Dehn WP solv lin $\leftrightarrow$ hyp. grp.

Q: what are the small cancellation properties of hyperbolic grps?

Theorem

The following conditions are equivalent for groups $G = \langle X \rangle$ with X finite.

- *There is a set of relators $R \subset A^*$ such that $G = \langle X \mid R \rangle$ and the Dehn algorithm of this presentation solves the word problem for G;*
- *The* **Dehn function** *of G (w.r.t any set of defining relators) is linear;*
- *Geodesic triangles in the Cayley graph $\Gamma(G, A)$ of G are uniformly slim (or uniformly thin);*
- *various other conditions.*

Idea?

does save save time ?

GAP Packages!

- Given fp hyp grp, calculate set of RW rules for Dehn alg

  KEY WORDS: R. Parker, C. Roney-Dougal

- KBMAG → can compute automatic grps & check if hyp (slower)

Q $O(WP(G)) = n^2$ which G's?

A: · Automatic grps & others.

· Cox $\sim n^{1+\varepsilon}$.

NOTE: · October → lectures ext next talk

· Nov → lectures by dev.