G fg, S finite gen set.

Growth fns:

$$V_{G,S}(n) = \#\{g : \ell_S(g) \leq n\}.$$

Rmk: $V_{G,S}(n+m) \leq V_{G,S}(n)\, V_{G,S}(m)$

$$V_{G,S}(n+1) = |S|\, V(n)$$

$$\exists \lim \frac{\ln V_{G,S}(n)}{n}$$

& $\lim_{n\to\infty} \sqrt[n]{V_{G,S}(n)} = v$

F_d, $d \geq 2$,

$$V(n+1) - V(n) \leq (2d)(2d-1)^n$$

Rmk: G admits a free subsemigrp

$\Rightarrow$ growth is exponential

We say that growth is poly

if $V(n) \leq C n^{\alpha}$

$\Longleftrightarrow$ G virt. nilpotent (Gromov).

If $V(n)$ is neither poly or exp,
then the growth is intermediate.

Gap Conjecture [Grigorchuk]

If $V(n)$ is NOT poly, then
$V(n) \geq \exp(C\sqrt{n})$ $\forall n$.

both possible

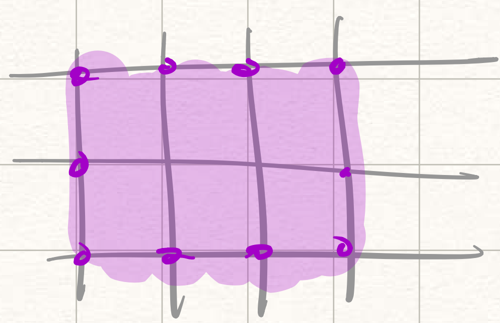
If G is residually nilpotent $\Rightarrow$ true

" — " residually solvable
$\Rightarrow V(n) \geqslant \exp(Cn^{1/6})$

Any fg grp G, if $V(n)$ is NOT
polynomial, then
$V(n) \geq n^{c \ln \ln^{c} n}$

$\exp(n^{a})$, $\exp(C \ln n \ln \ln n)$

Def: G is amenable (G fg)
if $\forall \varepsilon > 0$, $\exists$ finite $V \subset G$ st
$$\frac{\# \partial V}{\# V} \leq \varepsilon$$



where $\partial V = \{ v \in V \mid d_S(v, G \setminus V) = 1 \}$

Def (Følner fn) G, S,

$$\text{Føl}(n) = \min |V| \text{ st } \frac{|\partial V|}{|V|} \leq \frac{1}{n}$$

re-~~listen~~ rmk.

Thm (Coulon-Saloff-Coste)

$$\text{Føl}_{G,S}(n) \gg V_{G,S}(n)$$

asymp. larger.

(where: $f_1 \gg f_2$ if $f_2(n) \leq C_1 f(C_2 n)$
for some C_1, C_2.)

Pf: Take some V finite Følner set:

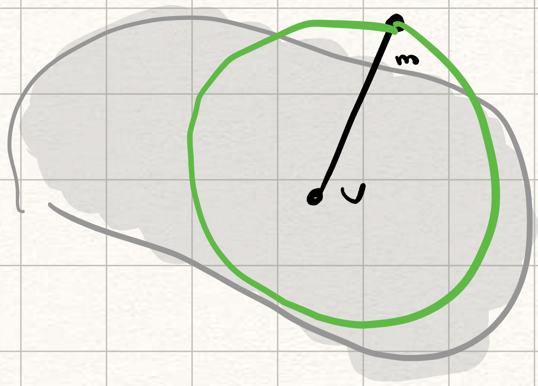
$$\frac{|\partial V|}{|V|} \leq \frac{1}{n}$$

fix minimal m:

$$V_{G,S}(m) \geq 2|V|$$

$$\Rightarrow V_{G,S}(n) \leq 2|S| \cdot |V|$$

($V(m+1) \leq |S| \cdot V(m)$ so if
$V(m) > 2|S| \cdot |V|$, $V(m+1) \geq 2|V|$



$\forall\, v \in V$

Take $B(v, m)$

e ?

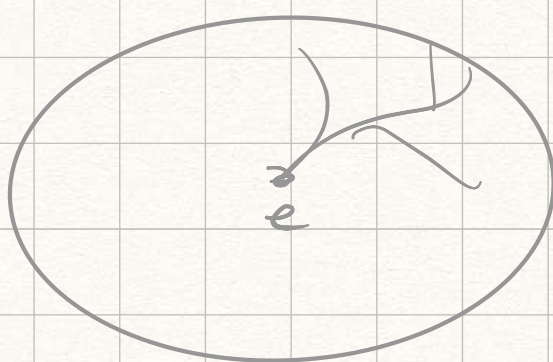
Let N be the number of pts outside of V w/ multiplicity.

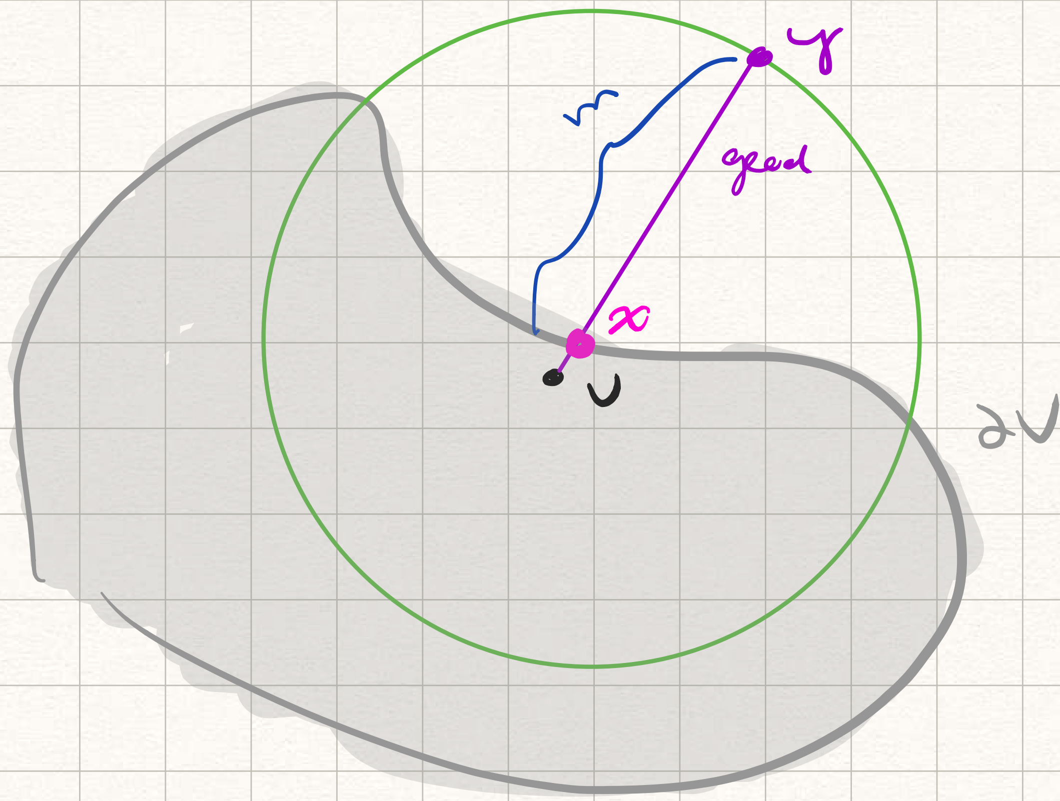
$$N \geq |V| \cdot (V_{a.s}(m) - |V|)$$
$$\geq |V|^2$$

Fix some spanning tree T

what. in $B(e, m)$ neither ? are geods

b/w x ?





If we know $v^{-1}y$, $\mathrm{dist}(x,y) \le m$,

so $N \le \underbrace{v(m)}_{\text{choices for } v^{-1}y} \cdot m \, |\partial V|$.

(re-listen)

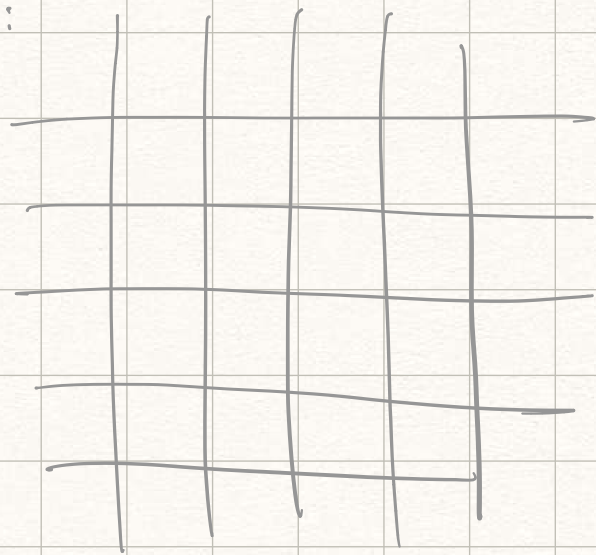
On the other hand,

$$|V|^2 \le N \le 2|S| \cdot |V| \cdot m \cdot |\partial V|$$

& $\dfrac{|\partial V|}{|V|} \geq \dfrac{\text{const}}{m}$.

$\square$

ex 1:



$G = \mathbb{Z}^d$

V is cube

$\partial V \sim C n^{d-1}$

work out.

$|V| = n^d$, $B(e,n) \sim n^d$

$\partial B(e,n) \sim C n^{d-1}$

ex 2: $V(n) \leq C n^d$

$\exists n$, $V(n+1) - V(n)$

$\leq \text{Const } n^{d-1}$

$$\frac{V(n+1)}{V(n)}$$

...

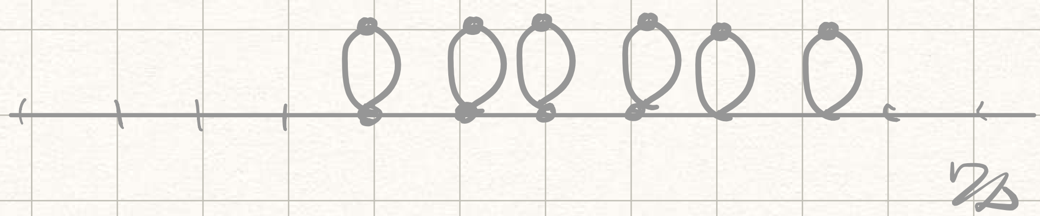
Pt: CS ineq opt for grps of poly growth.

What happens for exp?

<u>Recall</u>: Wreath product $A \wr B$

$$A \ltimes \sum_A B$$

<u>ex</u>: lamplighter grp $A = \mathbb{Z}$, $B = \mathbb{Z}_2$



C-S ineq is asympt. sharp for Lamplighter.

. some fn f acting on lamp.

$$|V| = 2^{2n+1}\,(2n+1)$$

$$\frac{|\partial V|}{|V|} \leq \frac{2}{2n+1}$$

<u>Gap Conjecture</u> (Grigorchuk, Pansu)

Any infinite fg G w/
$F\o l(n)$ is polynomial or
$F\o l(n) \geq \exp(Cn)$

Rmk: If growth is subexp then G amenable

Pf: subsequence of n,
$$\frac{\# \partial B(e,n)}{\# B(e,n)} \to 0$$

(if not: $V(n+1) \geq V(n)(1+c)\ \forall n > N$
$\Rightarrow V(n) \geq (1+c)^{n-ct}$)

Grigerchuck grp

Take rooted bin. tree.
Define recursive action.