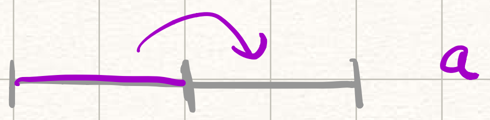
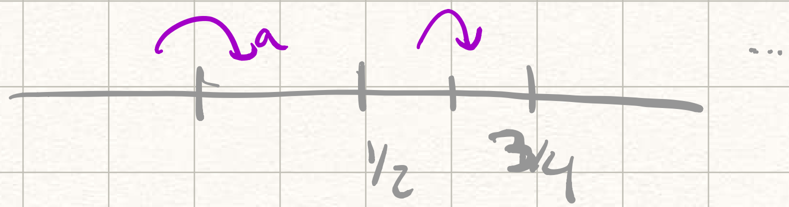
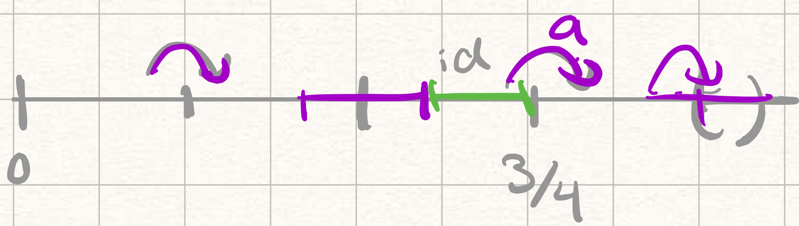
(Grigorchuk cont'd)



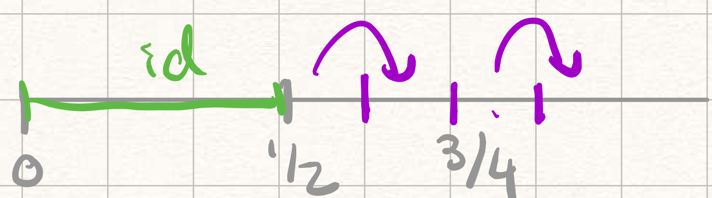


$b = (a, a, e, a, a, e, \dots)$



$c = (a, e, a, a, e, a, \dots)$

$d = (e, a, a, e, a, a, \dots)$



<u>Rmk</u>: $a^2 = e = b^2 = c^2 = d^2$

$d = bc,\ c = bd,\ b = cd$

$g \in G$, $\ell_S(\cdot) = \ell$,

then $g = (*)\,a\,(*)\,a \dots a\,(*)$

Lemma: $\exists\, \alpha < 1$, $V_{G,S}(n) \leq \exp(Cn^\alpha)$ $\forall n$.

Lemma: $\exists\, H$ fi subgrp in G, $S = \{a, b, c, d\}$

$\exists\, \varphi : H \to G^k$, injection

$\ell_{G^k} = \ell_S(g_1) + \ell_S(g_2) + \dots + \ell_S(g_k)$

$\exists\, \gamma < 1$, $\forall h \in H$ → the len in im is much shorter

$\ell(h_1) + \dots + \ell(h_k) \leq \gamma\, \ell_S(h) + \text{Const}$

where $\varphi(h) = (h_1, \dots, h_k)$.

Def: For virtually nilpotent N, an endomorphism $\psi : N \to N$, is a dilation if $\forall n \in N$,

$\ell_S(\psi(n)) \geq (1+C)\, \ell_S(n)$,

$\psi(N)$ of finite index in N.

ex: $\psi : z_1, \dots, z_d \to 2z_1, \dots, 2z_d$.

ex: $\begin{pmatrix} 1 & a & c \\ & 1 & b \\ 0 & & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & pa & p^2c \\ & 1 & pb \\ 0 & & 1 \end{pmatrix}$

$\ell_S \sim a + b + \sqrt{c}$.

Reformulation, $\varphi = \psi^{-1}$

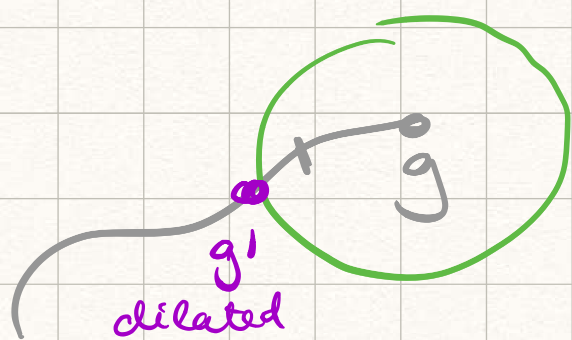
defines $\psi(N)$ fi subgrp

$$l_S(\varphi(n)) \le \gamma\, l_S(n).$$

Fran's lemma fg w/ dilation

$$\Rightarrow \text{growth}(n) \le n^d.$$

Pf:



$$\exists\, g' \in \psi(G)\, g$$
$$\in \psi(h)$$

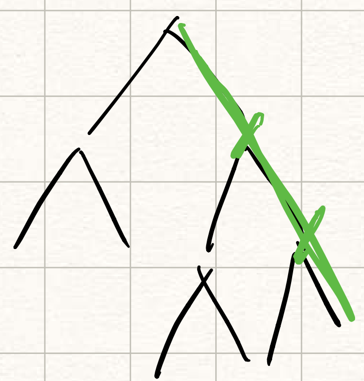
so $\ell(h) \le C\,\ell(g)$

$$\Rightarrow \quad V(n) \le C\,V(n\gamma)$$
$$\le C^2\,V(n\gamma^2)$$
$$\le C^3\,V(n\gamma^3)$$

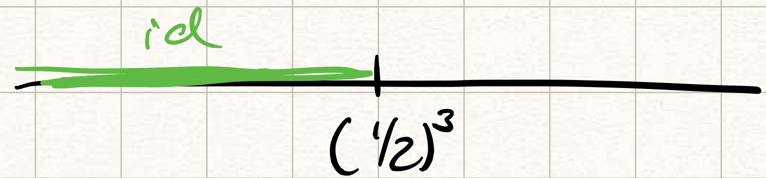
then $V(n) \le C^{\text{const}\,\ln n} \le n^d.$

look up pf Fran's lemma.

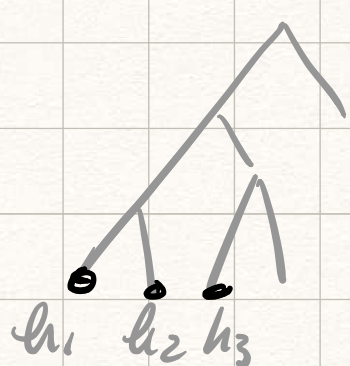
Back to Grigorchuk:



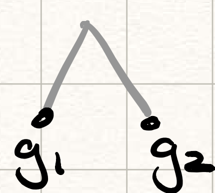
$H = \mathrm{Stab}(3)$ third level

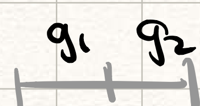


$\Psi : H \to G^k$
$h \mapsto (h_1, \ldots, h_k)$



$h \in H,\ h \in \mathrm{Stab}(1)$
$h = (g_1, g_2)$





$\ell_S(g_1) + \ell_S(g_2) \leq \ell_S(h) + ct$

$(*)a_1(*) + \dots \circ a(\underline{u})$ $\in \mathrm{Stab}(3) \subset \mathrm{Stab}(1)$

n_b be ~~the~~ numbers of b

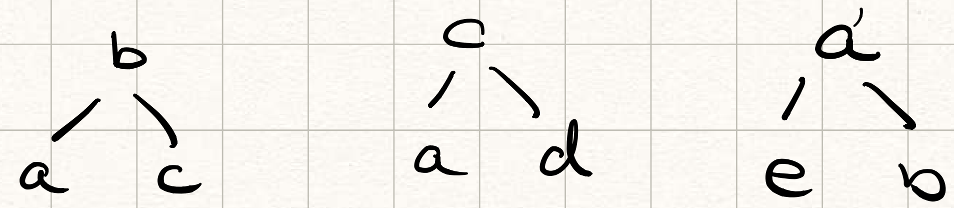
n_c c

n_a a

if $\dfrac{n_a}{n_b + n_c + n_d} \geq \dfrac{1}{3} \Rightarrow \ell(g_1) + \ell(g_2)$

$\leq \ell(h) + ct$

look $(*)\, a\, (*)$. $a(*)$



the total contr. of each $b - 2$

(of all gens b, c, d) $c - 2$

$d - 1$

$\ell(h) \sim 2(n_b + n_c + n_a \pm 1)$

$\ell(g_1) + \ell(g_2) \leq 2n_b + 2n_c + n_d$

either $\dfrac{n_a}{n_b + n_c + n_d} \geq \dfrac{1}{3}$

or $\dfrac{n_c}{n_b + n_c + n_a} \geq \dfrac{1}{3}$

or $\dfrac{n_b}{n_b + n_c + n_a} \geq \dfrac{1}{3}$

in any case,

$\ell(h_1) + \ell(h_2) + \dots + \ell(h_8)$

$\leq \frac{5}{6} \ell(n) + 3$

$\Rightarrow$ MAIN OBS. for subexp. growth.

Lemma 3 Let $f(n)$ be incr. fn.

st $\forall n > M$,

$f(n) \leq \text{Const} \, |\sum f(n_1) . f(n_2) \dots f(n_k)|$

$n_1, \dots, n_k,$

$n_1 + n_2 + \dots + n_k \leq \gamma n$, $\gamma < 1$,

$k \geq 2$

$\exists \alpha < 1$, $f(n) \leq \exp(\text{Const} . n^{\alpha})$

$\ell(h) = \ell$

$\varphi(h) = (h_1, \dots, h_k)$

$\ell(h_1) + \dots + \ell(h_k) \leq \gamma \ell + ct$

Pf of lemma 3 :

$\ln(f(n)) \leq An^{\alpha}$

we assume that this hold for any $n<m$.

each term satisfies the following:

$\ln(n_1) + \ln(n_2) + .. + \ln(n_k)$

$\leq n_1^{\alpha} + .. + n_k^{\alpha}$

$\leq (1-\varepsilon) A \cdot n^{\alpha}$

concavity

use α st:

$\left(K\left(\frac{\delta}{K}\right)^{\alpha} < 1 \right)$