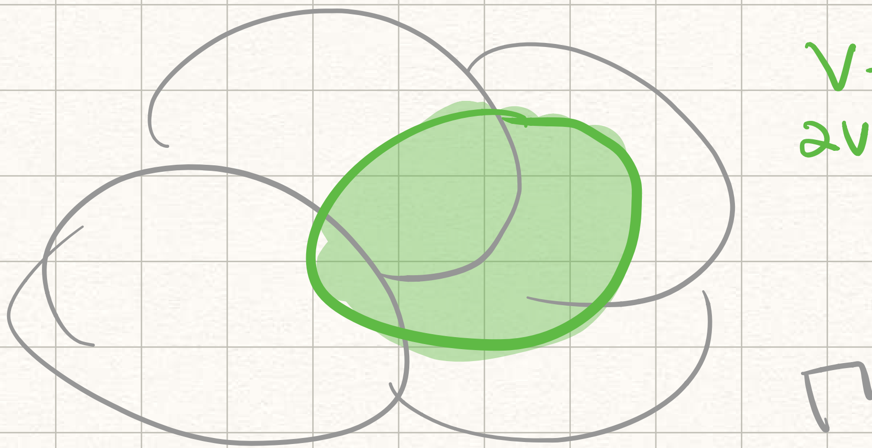
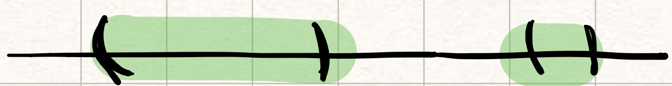
Isoperimetric inequalities



Isom peri prob: what can we say about V, ∂V?

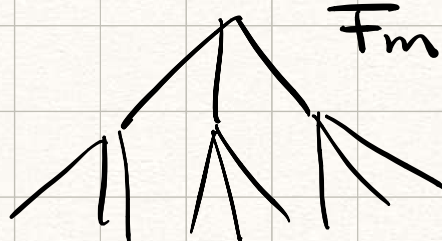
ex: $\mathbb{Z}$



$v > 1$

$\partial V \geq 2$

ex 2:



what are optimal sets for isoperimetric ineq?

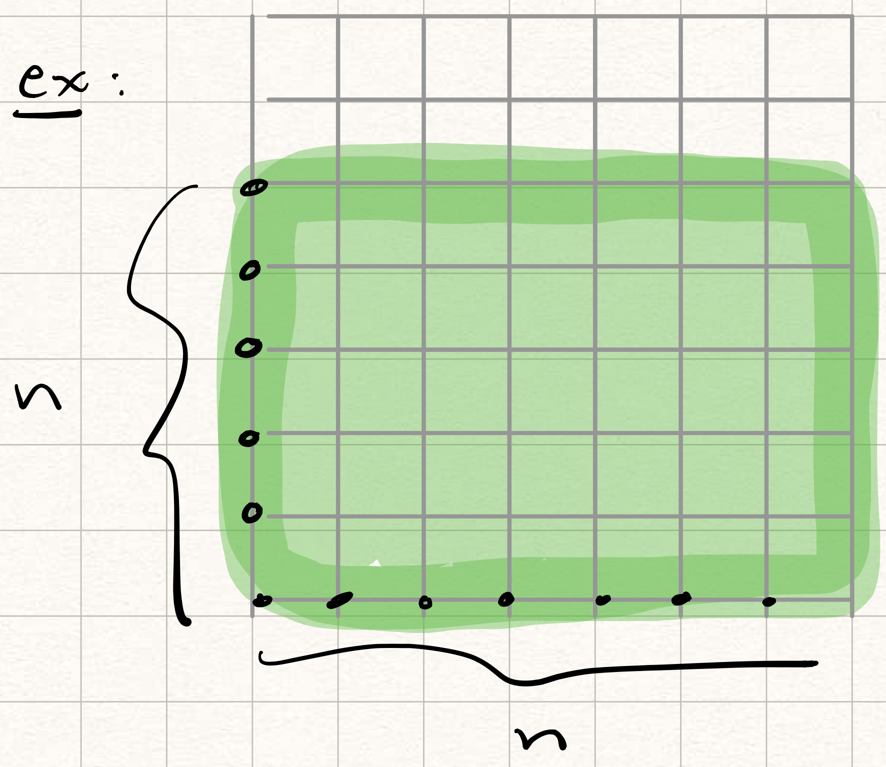
Def: G fg is non-amenable if $\exists C$ st $\forall V \subset G$, $\frac{|\partial V|}{|V|} \geq C$.

Let $V_{G,S}(n)$ be growth fn.

$$V(n) - V(n-1) = 2m(2m-1)^{n-1}$$

$$\Rightarrow V(n) = \sum_{i=0}^{n} 2m(2m-1)^{i-1}$$

ex: $\mathbb{Z}^2$

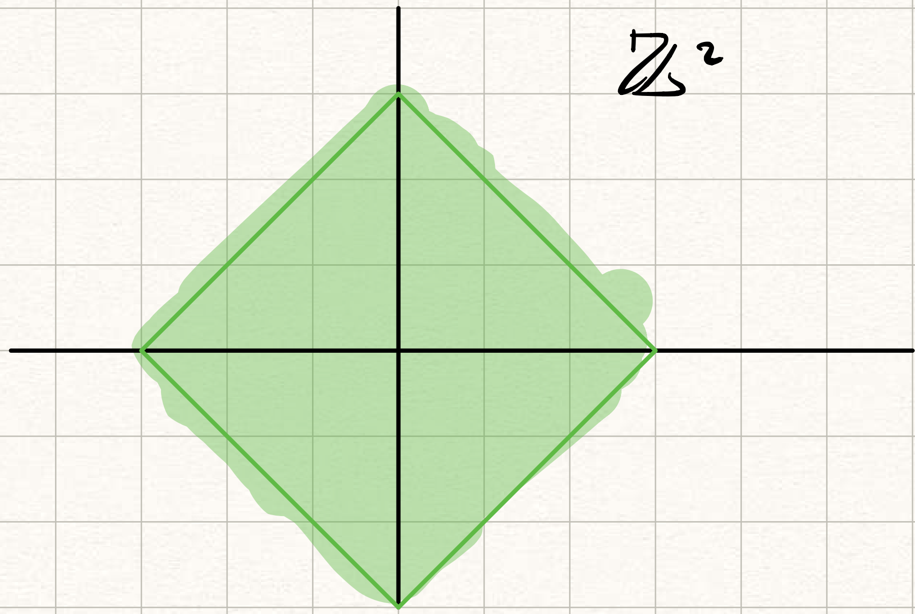


$V = n^2$

$\partial V = 4n$

$\left(\text{w/ gen set } S = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

Problem: NOT optimal set



$$\partial V = 4n,$$
$$V = (n+1)^2 + n^2$$

Pb: Balls are optimal in $\mathbb{Z}^2$

Lemma: Suppose $\dfrac{|\partial B(e,n)|}{|B(e,n)|} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow V(n) = |B(e,n)|$ is subexponential.

Pf: $$V(n) = \underbrace{\frac{V(n)}{V(n-1)}}_{\leq 1+\varepsilon} \; \underbrace{\frac{V(n-1)}{V(n-2)}}_{\leq 1+\varepsilon} \cdots V(1)$$

$$V(n) \leq (1+\varepsilon)^n \cdot \underline{C_\varepsilon}.$$

Lemma 2 : $V(n)$ is subexponential

$\Rightarrow \exists n_i \quad \frac{\# \partial B(e,n_i)}{\# B(e,n_i)} \to 0$

Open Q : $V(n)$ is subexponential, $\frac{|\partial B(e,n)|}{|B(e,n)|} \to 0$.

Thm : G virt. nilpotent

$\Rightarrow \underbrace{V(n) \sim C n^d \overset{?}{+} O(n^d)}_{\text{open}}$

Amenable & exponential growth

Amenability

$\exists V_i, \quad \frac{|\partial V_i|}{|V_i|} \to 0$

4 elementary operations which preserve amenability

- taking subgrps
- taking quotient
- taking extension
- $\bigcup G_i$, $G_i \subseteq G_{i+1}$.

G is **elmtary amenable** if it is obtained from finite Abelian or $\mathbb{Z}$ by 4 operations.

Prop: G is countable
$\Rightarrow$ G amenable $\Leftrightarrow$ all its fg subgrps are amenable.

Thm: $|G| = \infty$, fg, elmtary amenable.
- G polynomial growth

OR
- G exponential growth.

Wreath product

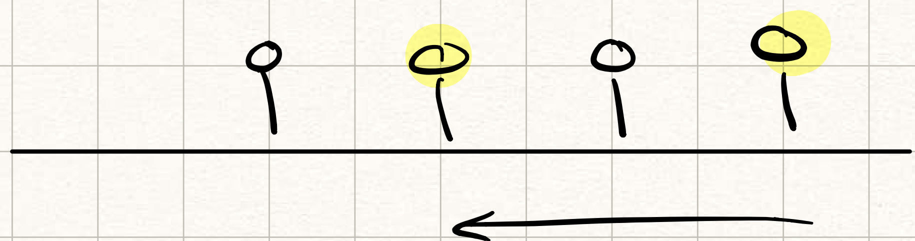
$$A \wr B = A \times \underbrace{\sum_A B}_{\text{an extension!}}$$

A acts on shifts.

$$\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \ltimes \sum_{\mathbb{Z}} \mathbb{Z}/2$$

has exponential growth.

ex: lamplighter grp $f: \mathbb{Z} \to \mathbb{Z}_2$.



$\mathbb{Z}$ acts by shifting

$$V(2n) \geq 2^n$$

$$\Omega_n = \left\{ (z,f),\ -n \le z \le n,\ \operatorname{Supp} f \subset [-n,n] \right\}$$

$$|\Omega_n| = 2^{2n+1} \cdot (2n+1)$$

Rephrased:

$$\text{Føl}(n) = \min V : \frac{|\partial V|}{|V|} \le \frac{1}{n}$$

Thm: G fg grp, $\forall C, \beta$ depending on G, S,

$$\text{Føl}(n) \geq V(Cn) + \beta$$

↑ is ∞ if G NOT amenable grps.

Gap Conjecture [Grigorchuck]

(Growth): G fg grp, either

- $V(n) \le Cn^d$

or

- $V(n) \geq \exp(n^a)$

for some a depending on n.

$a = 1/2 - \varepsilon$

(Strong form):

$$\text{Føl}(n) \le n^d$$

or $\geq \exp(n)$

Prop: G virt. nilpotent.

$$Cn^d \leq V(n) \leq Cn^d$$

Thm [Gromov] G is poly growth $\Rightarrow G$ virt. nilpotent.

$\hookrightarrow$ "polynomial growth thm"

3 Proofs:

- Asymptotic cones (Gromov)
- Harmonic fns (Kleiner)
- HFD (Ozawa)

Note: G nilpotent if

$$G \supset [G,G] \supset [G,[G,G]] \supset \cdots \supset 1$$

this eventually terminates.

OR: $\exists k$ in lower central series st $G_k = \{e\}$.

OR: $Z(G)$ center:

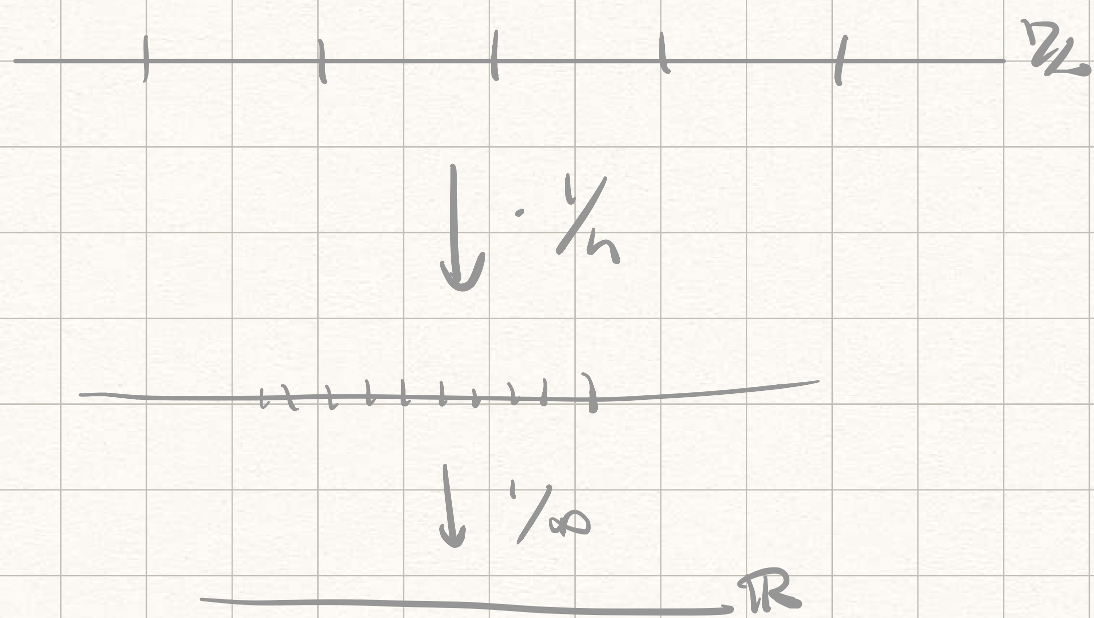
$$(G/Z(G))/Z(G/Z(G)) \cdots$$ terminate.

ex: Upper triangular matrices.

Idea: group commutators of different types
$\rightsquigarrow$ polynomially many.
$\rightsquigarrow$ Malcev.

words grow polynomially long (that's sort of the pf).

Asymptotic cone: what we see when you zoom out.



Nilpotent $\rightsquigarrow$ Lie grp.
take asympt. cone $\rightsquigarrow$ compact < Lie

<u>Note</u>: very specific for poly. growth.
(can't generalize that pf, but can generalize the others).