Exc: $H \leq G$ amenable

$\Rightarrow$ then H amenable.

Elementary amenable $\mathbb{Z} \wr \mathbb{Z}$

$\mathbb{Z} \wr \mathbb{Z}_2$

$\mathbb{Z}^d \wr \mathbb{Z}$

More exs of amenable:

- $BS(1,2)$ (just this one)

  exc: show it has exponential growth.

- Polycyclic
- Solvable

ex of non-elmtary but solvable:

$$\begin{pmatrix} 1 & * & * & & \\ & 1 & * & * & \\ & & \ddots & & * \\ 0 & & & 1 & \ddots \; a_{ij} \end{pmatrix}$$

w/ finitely non-zero $*$.

$j=1,\dots$

Send $A_i = \langle a_{ij} \rangle$, $A_i \mapsto A_{i+1}$

$G = \langle t, H \rangle$ st $tA_it^{-1} = A_{i+1}$

<u>ex</u>: $\mathrm{Sym}(\mathbb{Z})$ grp of syms of G.

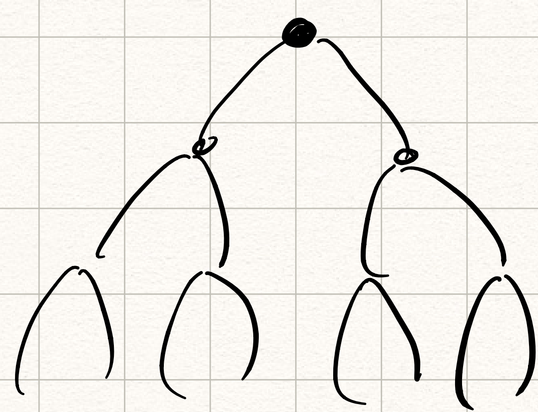
$G = \langle \mathrm{Sym}, \mathbb{Z}, + \rangle$ $\quad t: \mathbb{Z} \to \mathbb{Z}+1$

<u>Q</u>: G_1, G_2 nilpotent q-isom.

are G_1, G_2 commensurable?

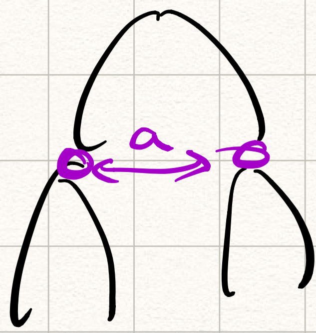
A: Not known.

<u>Grigorchuck</u> grp

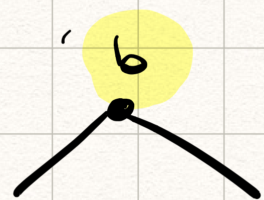


Gens: a, b, c, d

where



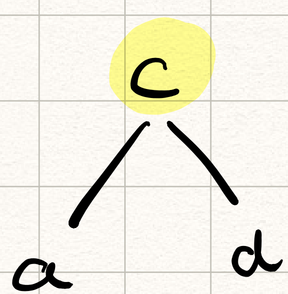
Action of b, c, d defined recursively



acts like a on 1st branch

acts like c on 2nd branch

sim.



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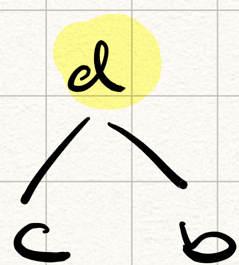
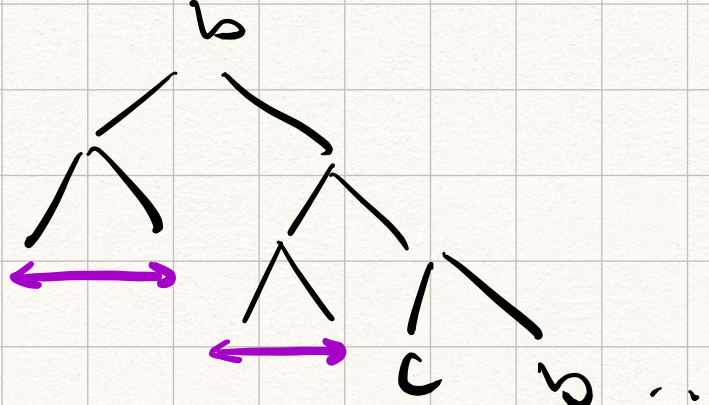


ie:



.. etc

$G := \langle a, b, c, d \rangle$ Grigorchuck grp.

Fact: Grigorchuck grp is grp w/ intermediate growth.

Lemma (MAIN): Garigor $a^2 = b^2 = c^2 = d^2 = e$,

b, c, d commute,

$a = bc$,

b, c, d generate $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Pf: $bc = cb$

$$bc \stackrel{?}{=} cb$$

first level ch

$a^2 \quad cd \qquad a^2 \quad dc$

2nd lvl will have to use induction.

Lemma 1 We want to prove

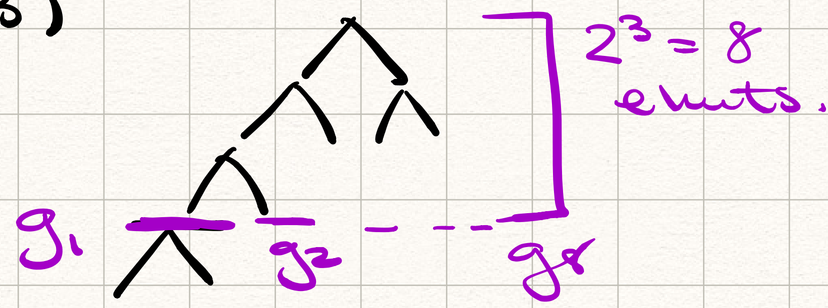
$$V(n) \le \exp(n^\alpha),$$

$$\alpha < 1,$$

$$\alpha = 0.76\ldots$$

Pf of main lemma:

$H = \mathrm{Stab}(3)$



$h \in H$, $h = (g_1, g_2, \dots, g_8)$.

Now,

$\ell_{GS}(g_1) + \dots + \ell_{GS}(g_8) \leq \lambda \ell_{GS}(h) + C$ ct

Lemma 2 G, H – subgrp of finite index in G

$H \to G^k$,

$h \mapsto g_1, \dots, g_k$, $\sum \ell(g_i) \leq \lambda \ell_{GS}(h) + C$ ct

$\lambda < 1$.

$\exists \alpha < 1$ depending on λ,

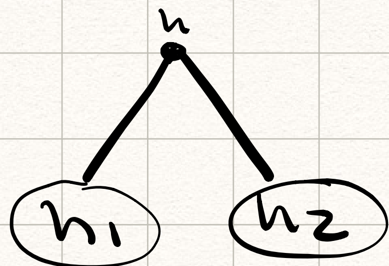
$V_{G,K}(n) \leq \exp(C \cdot n^{\alpha}) \cdot C_1$

Pf of Lemma 1 (weak form)

$h \in H$, $\ell_{as}(h) = n$.

$h = \star a \star a \star \ \dots \ \star$ where $\star = b, c, d$

$h \in \mathrm{Stab}(1)$



$\ell(h_1) = \# \star \le \frac{n}{2} + 1$

$\ell(h_2) = \# \star \le \frac{n}{2} + 1$

Now, $bc = d$, so $\overset{?}{n \,\vdots\, 2}$,

$\# \star \le n/2$

$\ell(h_1) + \ell(h_2) \le n$

$\ell(g_1) + \ell(g_2) + \dots + \ell(g_8) \le n$.

Let N_b, N_c, N_d be $\# \star$'s of b, c, d.

Then

$$N_d \geq \frac{1}{3}(N_b + N_c + N_a)$$

...

etc argue by level proportion...

$$\Rightarrow \ell(h_1) + \ell(h_2) \leq \left(1 - \frac{1}{6}\right) \ell(h)$$