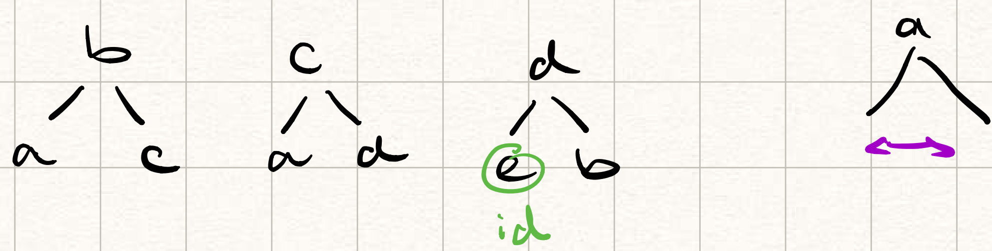
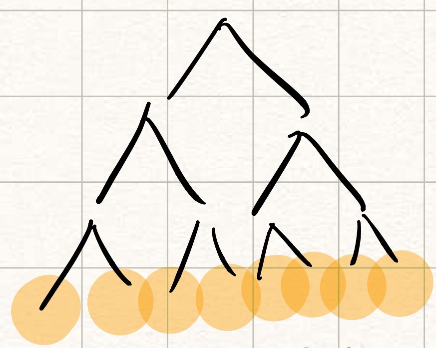
Goal: Get better contraction coefficient.

Grigorchuck grp: $G = \langle a,b,c,d \rangle$





$\ell = \sum \ell(y_i)$

We consider weights
$w : \{a,b,c,d\} \to \{A,B,C,D\}$

$g = s_1 \dots s_n,$
$s_i = \{a,b,c,d\}$

$$\ell_w(g) = \sum_i w(s_i)$$

$\mathrm{Stab}(1)$, $h = \mathrm{Stab}(1)$

$h = (g_1, g_2)$

$\ell(g_1) + \ell(g_2)$

$$\begin{array}{ccc} & da & \\ / & & \backslash \\ e & & b \end{array}$$

$a \longrightarrow$ doesn't change weight etc $\to A + D$

$b \longrightarrow$ we care $\;\triangle\; \Leftrightarrow \;\to B$

$\Rightarrow$ We would like

$$(A+C) \le \lambda (B+A)$$
$$(D+A) \le \lambda (C+A)$$
$$B \le \lambda (D+A)$$

w/ minimal possible λ.

Fix λ. $\Rightarrow$ 4 var, 3 linear eqs.

up to mult. ct

$\Rightarrow$ always have solutions.

Fix $A, B, C, D > 0$.

$$(A+C) = \lambda(B+A)$$
$$(D+A) = \lambda(C+A)$$
$$B = \lambda(D+A)$$

$$A+C = A + \lambda(D+A)$$
$$= A(\lambda+\lambda^2) + D\lambda^2$$

...

$$A := 1-\lambda^3$$
$$D := \lambda^3+\lambda^2-1$$
$$B := \lambda^3$$
$$C := \lambda^3+\lambda-1$$

$$B+A = 1$$
$$(A+C) = \lambda$$
$$C = \lambda - A$$

One last thing missing:

the relations $\Rightarrow \begin{cases} B \leq C+D \\ C \leq B+D \\ D \leq B+C. \end{cases}$

$\Rightarrow \lambda^3 + \lambda^2 + \lambda^2 - 2 \geq 0$.

minimal $\Rightarrow \lambda^3 + \lambda^2 + \lambda = 2$

2 complex roots,

1 real root λ_0,

$\lambda_0 \sim 0.8105\ldots$

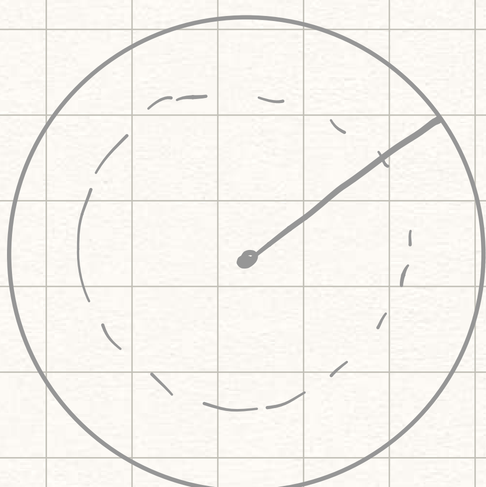
Lemma: $H \leq G$

$H \hookrightarrow G^d$, H finite index

$\lambda < 1$, $\ell(h) \leq \lambda \leq \ell(g_1) + ct$

$$\alpha = \frac{\ln d}{\ln d - \ln \lambda}$$

Growth of grp $\leq \exp(cn^{\alpha})$.



$H \cap B(e, r)$

how many h?

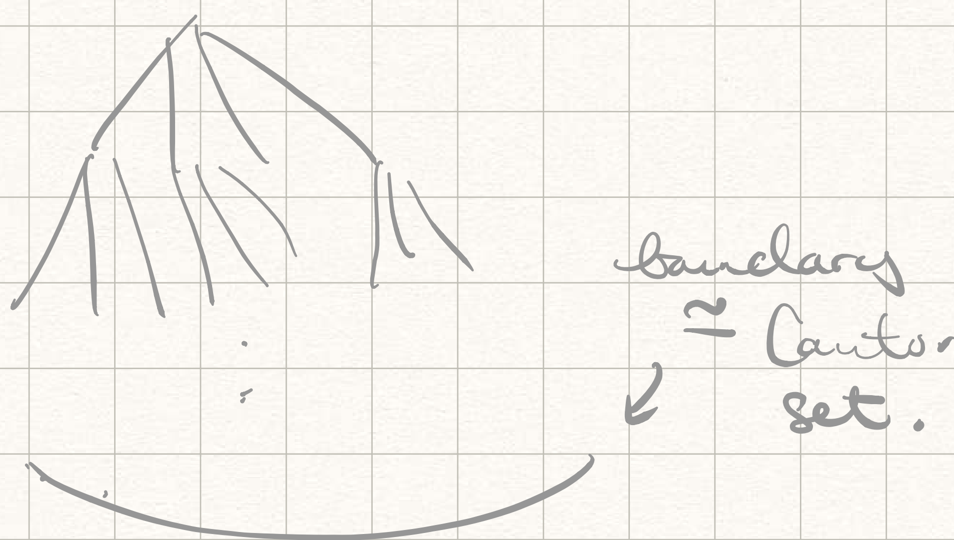
$h = (g_1, \ldots, g_d)$

Corollary : $\alpha_0 = \dfrac{\ln 2}{\ln 2 - \ln \lambda_0} \sim 0.767\ldots$

Open Q : Can we finitely pres grp of intermediate growth?

Rmk : First Grigorchuck grp grp is residually finite.

Thm : $\exists$ simple grps of intermediate growth.



Generalizations of Grigorchuk growth estimate

$G_{first} \geq \exp(n^{\alpha_0 - \varepsilon})$ $\forall \varepsilon > 0$.

[NE, Tangi, Zhang].

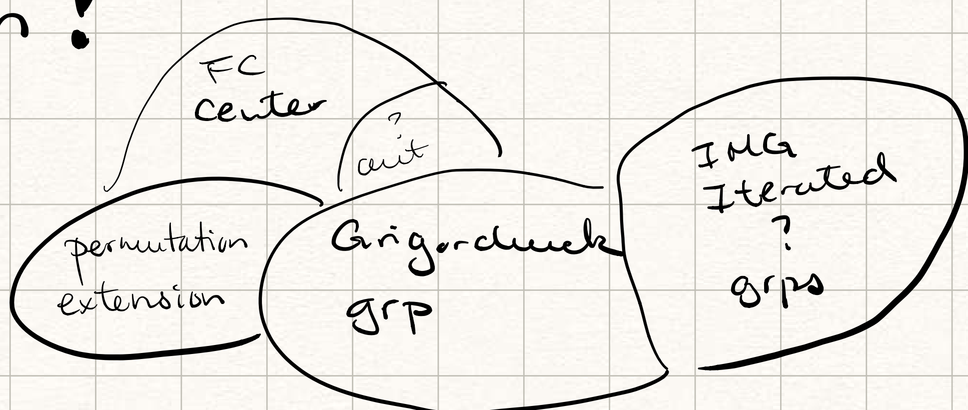
$f(n) \geq n^{\alpha_0}$ non-decreasing subadditive.

$\exists G$, $V(n) \sim \exp(f(n))$?

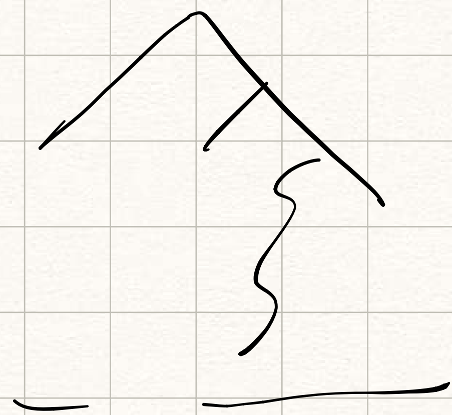
Rmk: $V(n+m) \leq V(n)V(m)$

$\ln V(n+m) \leq \ln V(n) + \ln V(m)$

Note: NOT every grp of intermediate growth admit contraction fn!



Note:

- 

Boundary at tree X, same orbit

$$(G_{Grig}, X) \sim \mathbb{Z}/2\mathbb{Z}.$$

- G_{Grig} actually admit Følner functions!

Exp growth	Subexp growth
Non-amenable	Amenable

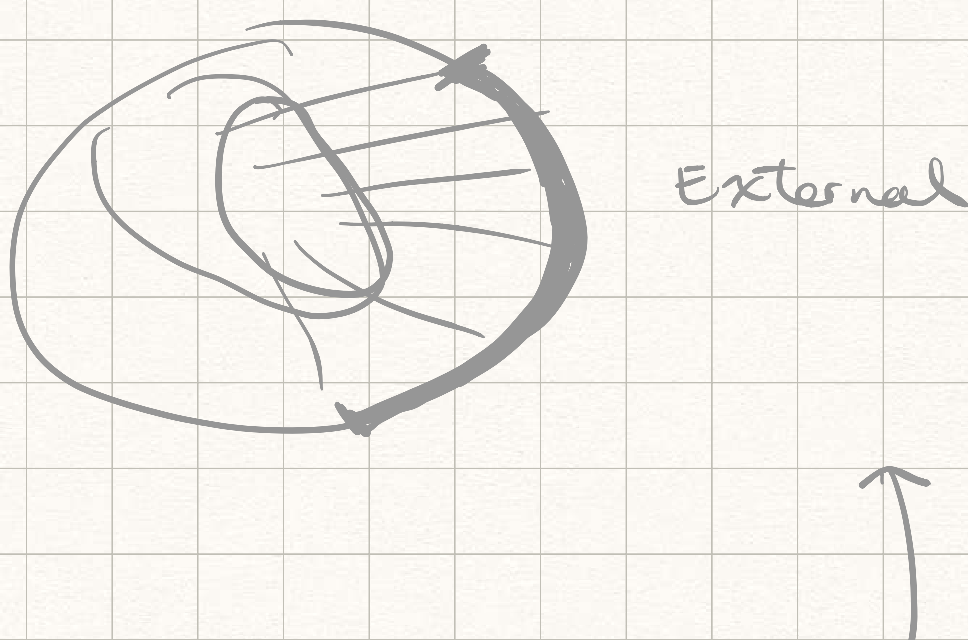
Q: G of exponential growth, does it admit a Lipschitz embedding of a binary tree?

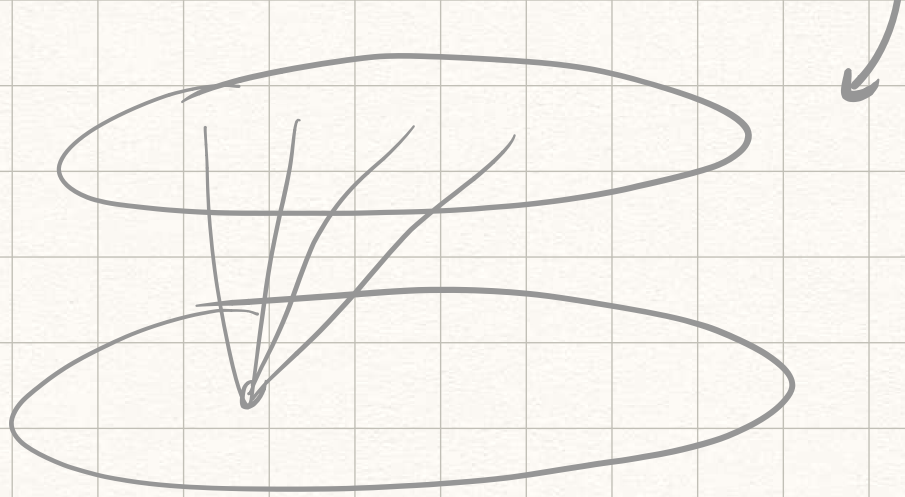
A: $\Longleftrightarrow$ Rosenblatt

Lemma: Inside each non-amenable grp, there is a tree!

Pf: $|\partial\Omega| \geq C|\Omega|$, up to changing gen set, $C \geq 2$.

Look at external boundary.





→ Bipartite w/ (2,1)-marriage

→ Get a tree ~~this~~ way.

<u>Open</u>: Grp of exp growth $\overset{?}{\Longrightarrow}$ tree ?