Arithmetic lattices

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- THE example: $\underbrace{SL_n(\mathbb{Z})}_{\text{this is a lattice in } SL_n(\mathbb{R})} \subset SL_n(\mathbb{R})$

  det = 1 is poly eq of deg n, $SL_n(\mathbb{Z})$ = int sol.

- $G \subset GL_n$ defined by polynomial eqs w/ integer coeffs.

  in the same way,

- $\underbrace{G(\mathbb{Z})}_{\text{int sols}} \subset \underbrace{G(\mathbb{R})}_{\text{real sol.}}$ ← thm: this is a lattice in $G(\mathbb{R})$

  (so SL_n analogy holds).

Careful: $SL_n(\mathbb{R})$ can be written in several ways as $G(\mathbb{R})$.

algebraic grp

ex: Quaternion algebra.

interesting G = what is the G you are interested in? to realize wanted matrix grp.

Thm (Margulis)

If $G(\mathbb{R})$ (Lie grp.) has $\underbrace{rk \geq 2}_{\text{dim of Cartan subgrp.}}$, then

$SL_n(\mathbb{R})$, $n \geq 3$, $SO(p,q)$, $SU(p,q)$

are the only examples. $p \geq 2, q \geq 2$.

orthogonal grp which act $x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$

preserving this norm

In $SO(2,1) = \mathrm{Isom}(\mathbb{H}^3)$,
you can deform lattices: the space of arithmetic lattices is countable.

ex of arithmetic lattices in $SL_2(\mathbb{R})$

① $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$

② co-cmpt arithmetic lattices

$$Q\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 - y^2 - \sqrt{2}\, z^2$$

$G = SO(Q) \subset SL_3$ } want solutions which respect:

Orthogonal := Mat = transpose
special: det 1.

$G(\mathbb{R})$? real solution to $Q\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$G(\mathbb{R}) = SO(Q,\mathbb{R}) \times SO(Q^\sigma,\mathbb{R})$$

can "place" $\sqrt{2}$ as $+\sqrt{2}$ or $-\sqrt{2}$, so depending on what you do get diff mats.

$$Q^\sigma = x^2 - y^2 + \sqrt{2}\, z^2$$

but $\cong SO(1,2,\mathbb{R}) \times SO(3,\mathbb{R})$

"compact factor" → don't care.

Integer pts sols? $(x, y, z \in \mathbb{Z})$

Mat in $\Gamma = SO(Q, \mathbb{Z}[\sqrt{2}])$

As the lattice takes coords

in $\mathbb{Z}[\sqrt{2}]$.

- Γ is a lattice in $G(\mathbb{R})$.
- Then $\pi(\Gamma) \subset SO(1,2,\mathbb{R})$ is a lattice (exc) → proj to first coord.

- It can be seen to be co-compact.

Some precision on number fields, places.

- k number field, finite ext $\mathbb{Q}$.
- k comes w/ finite number of places

  - $k \hookrightarrow \mathbb{R}$
    real places = set of embeddings in $\mathbb{R}$.

  - $k \hookrightarrow \mathbb{C}$
    complex places = " " in $\mathbb{C}$. which are NOT real.

Then r_1 denote # real places
$\sigma_1, \ldots, \sigma_{r_1}$ } real places (maps)

$2r_2$ # complex places

Hc $\sigma : k \to \mathbb{C}$ always

$\bar{\sigma} : \bar{k} \to \mathbb{C}$ come in pair

for complex.

$\sigma_{r_1+1}, \ldots, \sigma_{2r_2+r_1}$. complex places (maps)

Now if $G \subset GL_n$ defined by eqs w/ coefficients in k,

this must be a Cartesian prod.

forget about conj b/c want $G(\underline{\mathbb{R}})$ place the same.

$$G(\mathbb{R}) = \prod_{i=1}^{r_1} G^{\sigma_i}(\mathbb{R}) \times \prod_{j=1}^{r_2} G^{\sigma_{r_1+j}}(\mathbb{C})$$

matrices fulfilling the eqs, placed in many ways over reals.

where G^σ is the subgrp of $GL_n(\mathbb{R})$ (or $GL_n(\mathbb{C})$) of matrices fulfilling the im. under σ of the equations

<u>Recall</u>: For $SO(Q)$ before:

$$Q = x^2 + y^2 - \sqrt{2}z^2$$

$\swarrow$ two places $\searrow$

$$Q^{id} = x^2 + y^2 - \sqrt{2}z^2 \qquad Q^\sigma = x^2 + y^2 + \sqrt{2}z^2.$$

see this as sol w/ x,y,z.

see this as sol by placing $\sqrt{2}$ as $-\sqrt{2}$.

If O_K is the ring of integers

$$G(O_K) = \Gamma \xrightarrow[\star]{} G(\mathbb{R}) \qquad (\star)$$

$$\| $$

$$\left\{ \begin{array}{c} \gamma \in GL_n(O_K) \\ \text{fulfilling the} \\ \text{equations} \end{array} \right\}$$

$\star$: **thm** if your G grp is good enough, this image is a lattice in $G(\mathbb{R})$.

$$\gamma \mapsto (\underbrace{\gamma^{\sigma_1}, \gamma^{\sigma_2}, \ldots, \gamma^{\sigma_{r_1}}}_{\text{real places}}, \underbrace{\ldots, \gamma^{\sigma_{r_1+r_2}}}_{\text{complex places.}})$$

Now we focus on $PSL(2, \mathbb{C})$.

Def

Γ_1 and Γ_2 are **commensurable** if $\exists\, g \in PSL(2, \mathbb{C})$ st $g\Gamma_1 g^{-1} \cap \Gamma_2$ has finite index in both $g\Gamma_1 g^{-1}$ and Γ_2.

also:

$$\begin{tikzcd} & N \arrow[dl, "f_i"'] \arrow[dr, "f_i"] & \\ M_1 = \mathbb{H}^3/\Gamma_1 & & M_2 = \mathbb{H}^3/\Gamma_2 \end{tikzcd}$$

is equiv in this setting.

Def: An **arithmetic lattice** Γ in $PSL(2,\mathbb{C})$ is a subgrp commensurable to a grp Γ constructed like (*) where $G(\mathbb{R}) \cong PSL(2,\mathbb{R})$ compact factors.
But this is hard to check.

Def: An **arithmetic 3-orbifold** is a quotient $\Gamma \backslash \mathbb{H}^3$ where Γ is an arith. lattice.

If Γ has no torsion, then the quotient is an **arithm. 3-manifold**.

$$PSL_2(\mathcal{O}_d) \subset PSL_2(\mathbb{C})$$

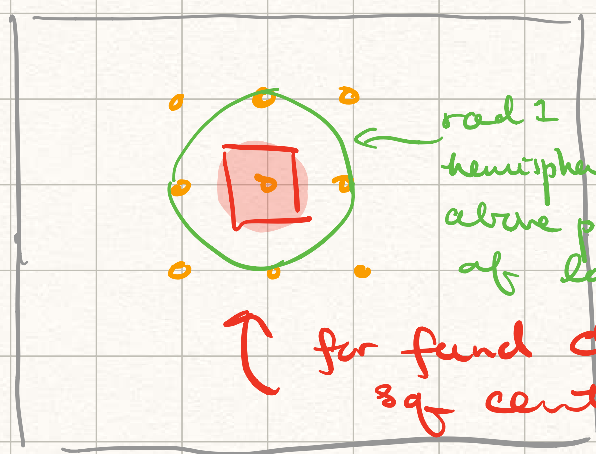
Ex: $PSL_2(\mathcal{O}_d)$ Bianchi grps where $\mathcal{O}_d$, ring of integers;

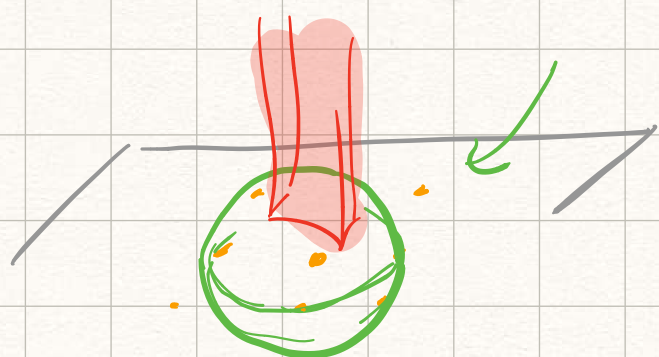
$\mathbb{Q}[\sqrt{-d}]$ has only one (pair of) complex plane.

Claim: Bianchi grp are non-cmpt arithmetic lattices.

will show constructed as (★)

Pf: $d=1$ let us construct a fund domain $SL_2(\mathbb{Z}[i])_\infty \supset \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$





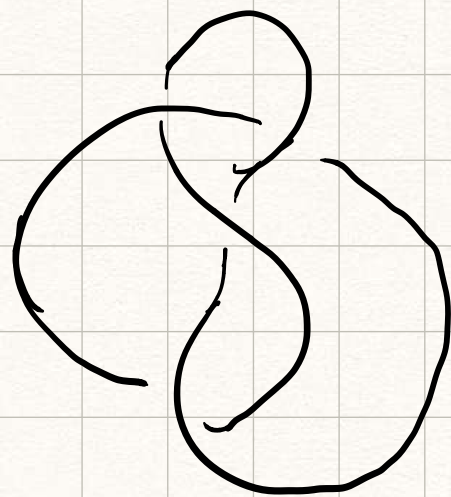
$$\mathrm{Stab}_\infty(\Gamma_1) = \left\langle \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & \\ & i \end{pmatrix} \right\rangle$$

Note that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in PSL(2, \mathbb{Z}[i])$

Similar pf for other Bianchi grp.

figure 8 knot complement

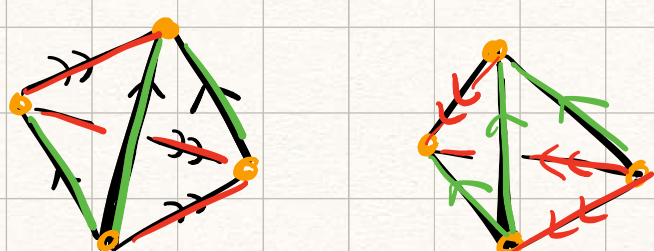
$S^3 \setminus K$ is a 3-manif

$K =$  $\subset S^3$.

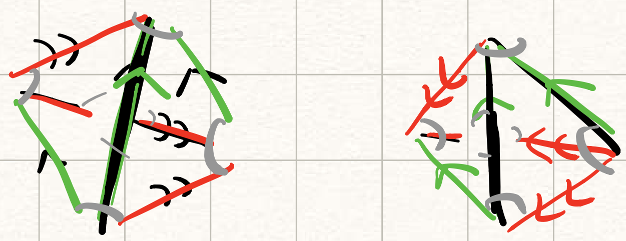
Thm (Riley)

→ It is a knot complement.
→ It is arithmetic.
→ It is isomorphic to an index 12 sub-grp of $PSL(\mathcal{O}_3)$.

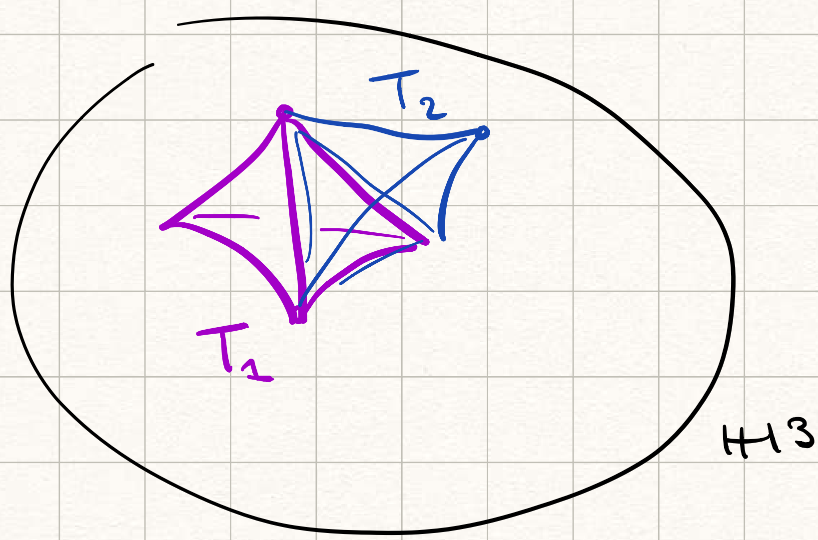
Pf: To see that, begin w/
$S^3 \setminus K$ = face pairings of



Obs: link of • (w/ arrows)
in torus.
⇝ makes not manifold
⇒ remove it.

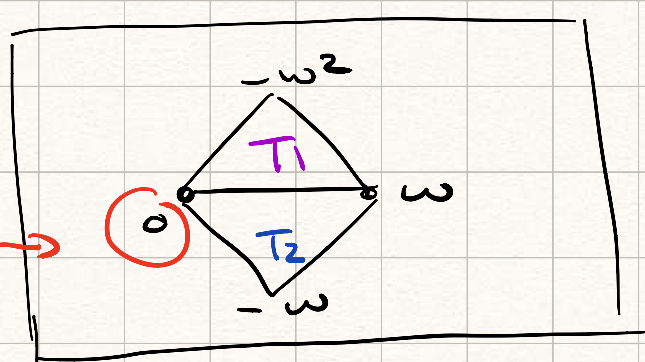


Now map it to $\mathbb{H}^3$



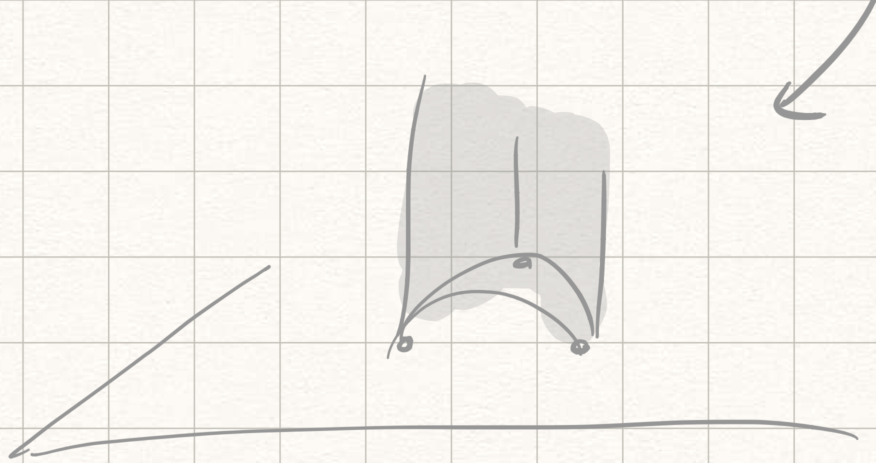
Fact:

ideal triang.



$$\omega = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$$

looks like this in 3D.



$T_1 \cup T_2$ is a fundamental domain for a rep

$$\rho : \pi_1(S^3 \setminus K) \to PSL(2, \mathbb{C})$$

You can write down the matrix giving the 3 remaining face pairings (they will be inside $PSL_2(\mathcal{O}_3)$).