Ref: MacLachlan - Reid.

Arithmetic 3-manif $\leftrightarrow$ Arithm of 3-manif

K field.

Def: A quaternion algebra over K, $K\langle 1, i, j, k\rangle$ w/ product def'd by

4 dim.

$$ij = -ji = k,$$
$$i^2 = a,\ j^2 = b,$$
$$\Rightarrow k^2 = -ab.$$

$a, b \neq 0$, $a, b \in K$ (ie in $K^\times$)

This alg is denoted $\left(\frac{a,b}{K}\right)$.

$$\left(\frac{a,b}{K}\right) \simeq \left(\frac{ac^2,b}{K}\right) \simeq \left(\frac{a,bc^2}{K}\right)$$

$$\simeq \left(\frac{b,a}{K}\right)$$

for any $c \in K^\times$,

and $\left(\frac{a,b}{K}\right) = \left(\frac{a,-ab}{K}\right)$. $(\star)$

ex: $K = \mathbb{C}$, there is only $A = \left(\frac{1,1}{\mathbb{C}}\right)$.

$$A \cong M_2(\mathbb{C})$$

$$1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \qquad k \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$i \mapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and this generalizes to any K:

$$\left(\frac{1,1}{K}\right) \cong M_2(K).$$

ex: $K = \mathbb{R}$.

$$\left(\frac{1,1}{\mathbb{R}}\right), \quad \overset{\text{same}}{\left\{ \begin{matrix} \left(\frac{-1,1}{\mathbb{R}}\right) \\ \updownarrow i \leftrightarrow j \\ \left(\frac{1,-1}{\mathbb{R}}\right) \end{matrix} \right\}}, \quad \left(\frac{-1,-1}{\mathbb{R}}\right)$$

$\left(\frac{1,1}{\mathbb{R}}\right) \cong M_2(\mathbb{R})$

by $\cancel{a}$ $b \mapsto -ab$

$$\left(\frac{1,1}{\mathbb{R}}\right) \cong M_2(\mathbb{R})$$

$\left(\frac{-1,-1}{\mathbb{R}}\right) = \mathbb{H}$

Hamilton's quaternion.

$i^2 = j^2 = k^2 = -1$.

Conjugation, Norm & trace

$$A = \left(\frac{a,b}{K}\right)$$

$$x = x_0 + x_1 i + x_2 j + x_3 k.$$

conj: $\bar{x} := x_0 - x_1 i - x_2 i - x_3 k.$

norm: $n(x) := x\bar{x}$

$$= x_0^2 - a x_1^2 - b x_2^2 + ab x_3^2$$

$$tr(x) = x + \bar{x}$$

for $M_2(K)$,

$det(x) = n(x).$

can show!

$$tr(x) = tr(x)$$

quat sense mat sense.

Rmk: $x^2 - tr(x)\,x + n(x) = 0$

(Cayley-Hamilton thm for $M_2(K)$).

Wedderburn thm

A quaternion alg. over K is
- either $M_2(K)$
- or a division alg.

Now take a quat. alg. over K. A.

Let

← grp of matrices of norm 1.

$$A^1 := \{x \in A \mid n(x) = 1\} \subset K^4 \xrightarrow{\sigma_i} \mathbb{R}$$

↑ which you can place in $\mathbb{R}$.

A^1 a mult. grp,

n is multiplicative.

$$A(\mathbb{R}) = \prod_{i=1}^{r_1} \left(\frac{\sigma_i(a), \sigma_i(b)}{\mathbb{R}} \right) \times \prod_{j=r_1+1}^{r_2} \left(\frac{\sigma_j(a), \sigma_j(b)}{\mathbb{R}} \right)$$

product is all possible places,

Recall:

K has $\sigma_1, \ldots, \sigma_{r_1}$ real places,

$$\left\{ \begin{matrix} \sigma_{r_1+1}, & \ldots & \sigma_{r_1+r_2} \\ \bar{\sigma}_{r_1+1} & & \bar{\sigma}_{r_1+r_2} \end{matrix} \right\} \text{ complex.}$$

NB: if $x \in A^1$, its image in any of these has norm 1.

This implies that $\Downarrow$

Now we have a morphism:

$A \to A(\mathbb{R})$

$$\Rightarrow A^1 \xrightarrow{\sim} \prod_{i=1}^{r_1} \left(\frac{\sigma_i(a),\sigma_i(b)}{\mathbb{R}}\right)^1 \times \prod_{j=1}^{r_2} \left(\frac{\sigma_j(a),\sigma_j(b)}{\mathbb{C}}\right)^1$$

and for each factor,

$$\simeq M_2(\mathbb{C})^1 \quad \text{by first pages.}$$
$$\wr\!\wr$$
$$SL_2(\mathbb{C}).$$

Now for the real factors, they are isom to

→ either $M_2(\mathbb{R})^1 = SL_2(\mathbb{R})$.

→ or $\mathcal{H}^1$. ← Hamiltonian quat of norm 1.

if $x \in \mathcal{H}$,

$$n(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2$$

$\Rightarrow$ as a grp, it's $SU(2)$

Q: Want lattice. What are integers in A^1?

Def (Integer ring)

$\mathcal{O}_K$ = roots$^{\in K}$ of monic poly w/ coeffs in $\mathbb{Z}$. $\mathbb{Z} \subset \mathcal{O}_K$ always.

$x \in A$ is an integer.

if $\begin{cases} tr(x) \\ n(x) \end{cases}$ are in fact in O_K

Recall our Cayley-Hamilton eq:

$x^2 - tr(x)x + n(x) = 0$ ⇒ sols in O_K,

but. ⚠ This does not form a ring.

ex: not closed in product.

→ Need to define an "order on A".

Order on A

- fg O_K submodule of A,
- generating A as a K-vect.
- is a unital ring.

ex: $\underbrace{O_K[1, i, j, k]}_{\text{fg, has } 1\text{, gens } A\ ✓}$ w/ $a, b \in O_K$.

Now, $O^1 \subset A^1$ is a discrete subgrp for any order O.

$O^1 = O \cap A^1$

Point: $\mathcal{O}^1 \hookrightarrow A^1 \hookrightarrow A^1(\mathbb{R})$

$\uparrow$

the image is a lattice.

How to construct lattices in $SL_2(\mathbb{C})$?

$\curvearrowright$ prods of $SL_2(\mathbb{R})$, $SL_2(\mathbb{C})$ & $SU(2)$.

$$A^1 = \underbrace{\prod_{i=1}^{r_1} \left(\frac{\sigma_i(a), \sigma_i(b)}{\mathbb{R}}\right)^1}_{SL_2(\mathbb{R}) \vee SU(2)} \times \underbrace{\prod_{j=1}^{r_2} \left(\frac{\sigma_j(a), \sigma_j(b)}{\mathbb{C}}\right)^1}_{SL_2(\mathbb{C})}$$

the only way: $SL_2(\mathbb{C}) \times$ compact factors

$r_2 = 1$ (K has only one complex place) so no other $SL_2(\mathbb{C})$ factor.

For real places,

$$\left(\frac{\sigma_i(a), \sigma_i(b)}{\mathbb{R}}\right) \simeq \mathcal{H}$$

must be $SU(2)$ and not $SL_2(\mathbb{R})$. for compact fact

"A ramifies at real places".

Thm: For K, A as ($\star$), the im of

$$\mathcal{O}^1 \hookrightarrow A^1(\mathbb{R}) \xrightarrow[\text{prj}]{} SL(2, \mathbb{C})$$

is a lattice in $SL(2, \mathbb{C})$.

what we were saying before!!.

Moreover, it is co-compact $\Leftrightarrow$ $A^1(\mathbb{R})$ has compact factors.

This is a complete descript of comm'able classes in $SL_2(\mathbb{R})$.

Rmk: → two lattices given by NOT (K, A) are NOT commensurable.

went fast don't worry

→ non-cocompact ⇒ K has only $\underline{1}$ complex place, and no real place!

Point:

⇒ the only possibility will be the Bianchi grp. (for wanted lattice in $SL(2,\mathbb{C})$).

This construction gives all commensurability classes of arithmetic lattices in $SL(2,\mathbb{C})$.

But lots of lattices as prod of $SL_2(\mathbb{R}) \times SL_2(\mathbb{C})$. b/c:

$$A^1 = \prod_{i=1}^{r_1} \left(\frac{\sigma_i(a), \sigma_i(b)}{\mathbb{R}}\right)^1 \times \prod_{j=1}^{r_2} \left(\frac{\sigma_j(a), \sigma_j(b)}{\mathbb{C}}\right)^1$$

Given a number field K, the construction of an infinite number of non-comm'able arithmetic lattices over K.