Reminder: A subgrp $\Gamma < PSL_2(\mathbb{C})$ is arithm. if it is commensurable to $\mathcal{O}^1$, where $\mathcal{O}$ is an order of a quat. alg. over K w/

$A(\mathbb{R}) \cong \mathbb{H}$

- K has only one complex place
- A "ramifies" at real place

Q: take Γ any lattice at $PSL(2,\mathbb{C})$. Can I decide if its arithmetic or not?

A: Yes. B/c to ANY lattice, there is an associated field and a quat. alg.:

$\rightsquigarrow$ Invariant trace field, quat. alg.

"Arithmetic OF lattices".

Given Γ, $\Gamma^{(2)} = \langle \gamma^2, \gamma \in \Gamma \rangle$

Fact: $\Gamma^{(2)} < \Gamma$ is normal and of finite index.

$k\Gamma := \mathbb{Q}[\underline{tr(\gamma)}, \gamma \in \Gamma^{(2)}]$

field gen by tr

Thm: ($k\Gamma$ w/ $\Gamma^{(2)}$ is an inv of comm.)
$k\Gamma$ is a number field and if Γ and Γ' are commensurable, then $k\Gamma = k\Gamma'$.

$\to$ it is computable.

It comes from the tr rel. in $SL_2(\mathbb{C})$:

$$tr(A)tr(B) = tr(AB) + tr(A^{-1}B)$$

$\to$ can actually find a finite subset of $\gamma \in \Gamma^{(2)}$ generating $k\Gamma$. If $\gamma_1, \ldots, \gamma_n$ are gens, take all

$$\gamma_i, \ \gamma_i\gamma_j, \ \gamma_i\gamma_j\gamma_k, \ldots$$

$$A\Gamma = k\Gamma[\gamma, \gamma \in \Gamma^{(2)}] \subset M_2(\mathbb{C}).$$

Thm: $A\Gamma$ is a quat. alg. over $k\Gamma$ and if Γ, Γ' are comm'able, $A\Gamma \simeq A\Gamma'$. ★ being comm'able to $\mathcal{O}^1 \Rightarrow$ arithm.

One can actually show that

$A\Gamma = k\Gamma[1, g, h, gh]$ where $g, h \in \Gamma^{(2)}$ w/ g, h acting irreducibly

→ don't gen an elmtary subgrp in $PSL_2(\mathbb{C})$.

ex.: take two hyp elmt w/ no common fixed pt.

Rmk: if $A \in SL_2(\mathbb{C})$,

$$A^2 = tr(A)A - Id$$

$$\Rightarrow tr(A^2) = tr(A)^2 - 2.$$

then $tr(A^n) = P(tr)$,

w/ P monic poly w/ $\mathbb{Z}$-coeff

w/ g, h as above.

$$A\Gamma = \left(\frac{tr(g)^2 - 4,\ tr[g,h]}{k\Gamma}\right)$$

Thm: If Γ is arithmetic "coming from" (K, A), then

grp field & quat alg. field are the same $\rightsquigarrow$ $k\Gamma = K$

$A\Gamma = A$. $\leftarrow$ grp alg & alg are the same.

Rmk: if Γ is arithm. over (K, A) w/ $K \subset \mathbb{C}$, $A \hookrightarrow A_{\mathbb{C}} \subset M_2(\mathbb{C})$.

then for any elmt $a \in A$,

$$\mathrm{tr}_A(a) = \mathrm{tr}_{M_2(\mathbb{C})}\, \rho(a)$$

$$\Rightarrow \mathrm{tr}_{M_2(\mathbb{C})}(\rho(a)) \in K$$

$$\Rightarrow k\Gamma \subset K.$$

Moreover, if $a \in \mathcal{O}$ an order,

$\Rightarrow \mathrm{tr}_A(a) \in \mathcal{O}_K$ $\leftarrow$ ring of int. of K,

We <u>almost</u> have $\mathrm{tr}(\gamma) \in \mathcal{O}_K$.

In fact, we only have $\mathrm{tr}(\gamma)$ is an alg. int for any $\gamma \in \Gamma$.

<u>B/c</u>: for some n, $\gamma^n \in \rho(\mathcal{O}^1)$ for $\mathrm{tr}(\gamma^n) \in \mathcal{O}_K$.

$\Rightarrow$ $tr(\gamma)$ is a root of $P(tr(\gamma)) - tr(\gamma^n)$ which is monic w/ O_K - coeffs.

Identification thm

$\Gamma < PSL_2(\mathbb{C})$ is arithm

$\Leftrightarrow$
- $k\Gamma$ has a unique complex place
- $A\Gamma$ "ramifies" at every real places.
- $tr(\gamma)$ is an alg. integer for $\gamma \in \Gamma$.

Moreover if $tr(\gamma) \in O_{k\Gamma}$ $\forall \gamma \in \Gamma$, then :

- Γ subgrp of $\rho(O^1)$, O an ord. of $A\Gamma$.

Rmk: The third cond. is granted by a finite number of $tr(\gamma)$ alg. integer.

Arithmeticity of knot complement

Thm (Reid) the 8-knot complement is the only arithmetic knot complement.

⇝ they just mean realize mat w/ pt. coords.

Sketch Pf: M arithm. knot complement.

⇝ commensurable to a Bianchi grp.

⇝ cusps in the Bianchi grp they should all be equiv

⇒ "the class number of K is 1"

⇝ finite list.

(In fact "cuspal coho. prob. Only a finite number of Bianchi grps contain link complement in S^3".)

In this case, only 1 order in $M_2(K)$
$$\overset{\shortparallel}{M_2(\mathbb{Z}\sqrt{-d})}.$$

$\rightsquigarrow$ homology arguments
$\Gamma \subset SL_2(\mathbb{Z}\sqrt{-d})$.

$\gamma \in \Gamma$ fixing ∞ a meridian,
γ normally generates Γ.

$$\gamma = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$$

Claim : x is a unit of $\mathcal{O}_d$.
if not, P prime $| \, x\mathcal{O}_d$,
$\gamma \in \Gamma_P = \ker(\Gamma \to PSL_2(\mathcal{O}_d/P))$.

$\Gamma \subset \Gamma_P$ $\rightsquigarrow$ it is an ex of a "principal congruence subgrp"

but: congruence subgrps have inequivalent cusps.

Very few units in $\mathcal{O}_d$, in fact, included in $\{\pm 1, \pm i\}$ if $d \neq 3$,

$d = 3$: $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

take the elmt in the stab. of 0, conj'd in Γ to γ,

$\delta = \begin{pmatrix} 1 & \\ y & 1 \end{pmatrix}$, y also a unit,

but $\operatorname{tr}(\gamma\delta) = 2 + xy$

$\operatorname{tr}([\gamma, \delta]) = 2 + (xy)^2$

for <u>all</u> choices in $\{\pm 1, \pm i\}$.

You construct a prod of γ, δ w/ tr 1 $\rightsquigarrow$ elliptic.

So $d = 3 \rightsquigarrow \Gamma_\infty \subset \Gamma$ but Γ_∞ is a max'al subgrp $PSL_2(\mathbb{C})$ w/o torsion. $\square$

Commensurability, et al.

Thm (Margulis) $\Gamma < PSL_2(\mathbb{C})$ lattice
is arithm. $\iff$ Comm(Γ) is dense
Commensurator

$$\mathrm{Comm}(\Gamma) = \left\{ g \in PSL_2(\mathbb{C}) \mid \underset{\Gamma}{\overset{\text{finite}}{\wedge}} (g\Gamma g^{-1}) \cap \Gamma \underset{g\Gamma g^{-1}}{\overset{\text{finite}}{\wedge}} \right\}$$

all the g's which cause commensurable subgrps.

$(\Rightarrow)$ $\Gamma = \rho(\mathcal{O}^1)$ order in A,
then any $g \in \rho(A^\times)$ belongs to
the Comm(Γ).

exc: $PSL_2(\mathbb{Q}) = \mathrm{Comm}(PSL_2(\mathbb{Z}))$

Thm: (Reid, Marg. M)

- If M arithmetic, then M contains 0 or infinitely many immersed totally geodesic surfaces.

- If a 3-manif M has ∞'ly sub surfaces, then it is arithmetic.