Groups as Geometric Objects

Group Presentation

$F(X)$ free grp of X. Then it satisfies the universal property

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, hook, "i"'] & G \\ F \arrow[ur, "\exists \text{ unique } \tilde{\varphi}"'] & \end{tikzcd}$$

= way of completing this

Every G is a quotient of a free grp, any grp satisfying the relations of G is a quotient.

ex: $G = \langle x, y \mid xy^{-2}, x^2y^{-3} \rangle = 1$

NOT obvious!

Higman Grps (source of counter ex)

$$H_n = \langle x_1, \dots, x_n \mid [x_i, x_{i+1}] = x_{i+1} \mod n \rangle$$

on ~~the~~ index

ex: $H_2 = \langle x_1, x_2 \mid [x_1, x_2] = x_2 \rangle$

$[x_2, x_1] = x_1$

$$H_4 = \langle x_1, \dots, x_4 \mid [x_1, x_2] = x_2,\ [x_3, x_4] = x_4,\ [x_2, x_3] = x_3,\ [x_4, x_1] = x_1 \rangle$$

$n = 2, 3 \Rightarrow H_n$ trivial

↑ hard exercise

$n \geq 4$. $F_2 \hookrightarrow H_n$

• H_4 has no finite index subgp

OPEN : H_4 has no amenable ∞-quotient.

hard ex: H_4 acts on a CAT(0) square cmplx w/ soluble stabilizers.

Fundamental Problems

- Dehn's.
- Burnside problem for every n.

Is there a G a non-trivial fg grp in which every element satisfies $x^n = 1$.

Q: Is such a grp always finite?

A: For $n \ggg 0$, there are ex's

↑ of ∞-grps.

By small cancellation theory.

Q: Does there exists a fp simple grp?

A: At least 1 → Burger-Moss ex's.

worth looking at pf.

Thm: A grp is fp
⇔ G acts on a simply connected, path connected (geodesic) topological space st the action is proper & co-compact.

Q: Is thm equiv of having Cayley graph?

A: Not equiv ⇝ what's an ex?

$G = fg \rightsquigarrow$ Cayley graph.

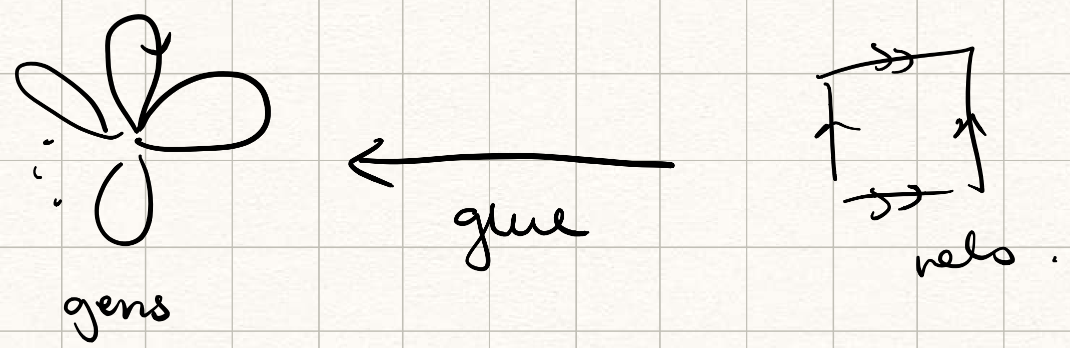
- is this thm useful b/c get action, and then know fg?
- can you construct fp using the action? A: ($\nexists$)

$\underbrace{\mathbb{Z}^2}_{G} \curvearrowright \mathbb{R}^2$

wouldn't

Cayley graph builder

know could still recover pres.



$$\tilde{P}^{(1)} = \mathrm{Cay}(G,S)$$

We call $\tilde{P}$ the presentation 2-complex or Cayley complex.

Def: A grp is CAT(0) if $\exists$ a CAT(0) space X on which G acts properly & co-compactly.

Hyp. space : "Thickened tree".

Def hyperbolic grp

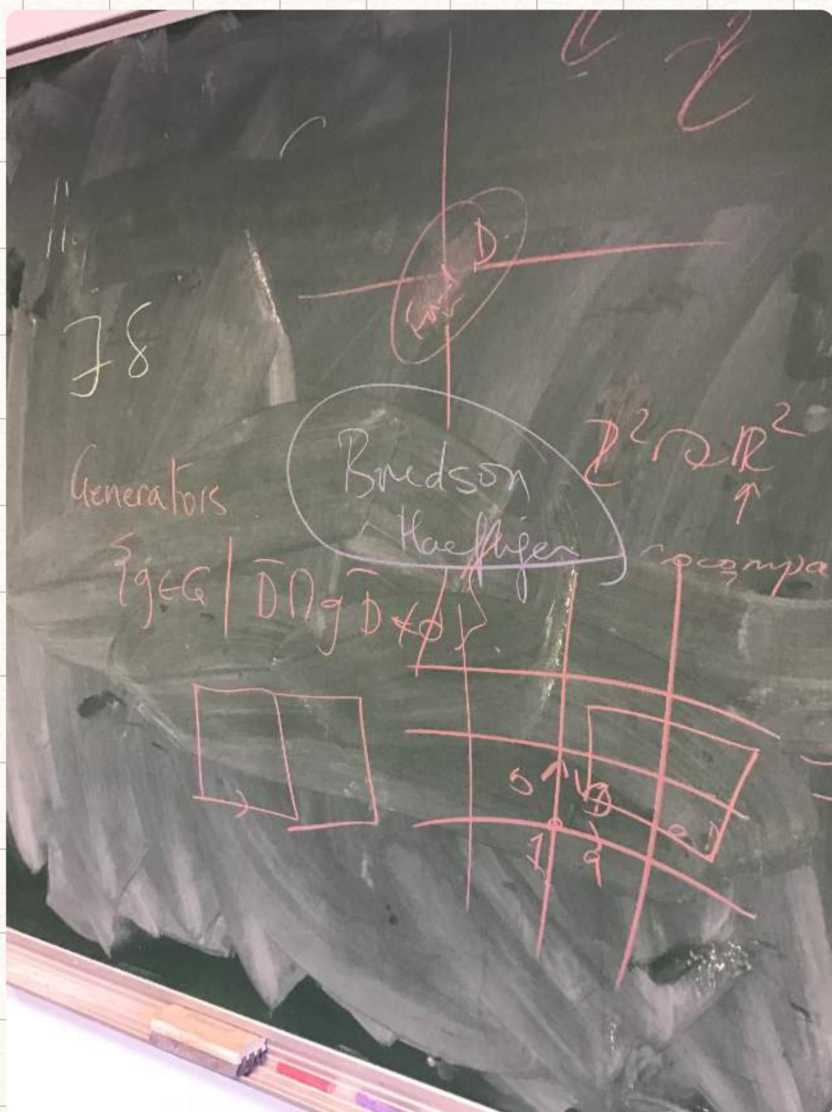
(i) Grp hyp @ prop + co-compact on hyperbolic space.

OR

(ii) Cayley graph is hyperbolic.

For many cases, this is NOT good enough..

$\rightsquigarrow$ there are <u>better</u> topo approx of the grp.



(A) : Draw fund domain, F then look at intersection of $\Gamma \curvearrowright F$

$\curvearrowleft$ action = proper

$\Rightarrow$ intersection is finite

= finite gen. set

\+ finite 2-cells.

ex: $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$.

Note : CAT(0) $\leftrightsquigarrow$ hyp not clear.