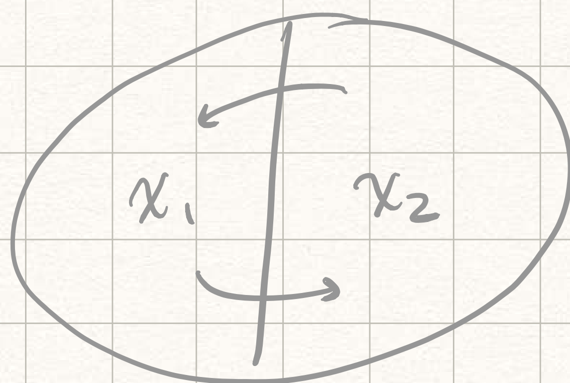
Ping Pong Lemma

Ping Pong Lemma

Let G grp acting on X. Let $A, B \leq G$.
Let $H = \langle A, B \rangle$.
If $\exists$ non-empty $X_i \subseteq X$, $i = 1, 2$
st $X_2 \not\subseteq X_1$ and

- $a X_1 \subseteq X_2$
- $b X_2 \subseteq X_1$

$\forall a \in A \setminus \{1\}$,
$\forall b \in B \setminus \{1\}$



Then $H \cong A * B$.

Free Products

Suppose A, B grps given by

$$A = \langle X \mid R \rangle$$
$$B = \langle Y \mid S \rangle$$

Then $A * B = \langle X \cup Y \mid R \cup S \rangle$

In particular, $A, B \cong \mathbb{Z}$, X, Y basis, $R, S = \emptyset$.

Then $A * B = F_2$ free on two gens.

Thm: A grp F is free w/ basis X
$\Leftrightarrow$ F acts freely on a tree.

($\Leftarrow$) $F \backslash T = \Gamma$ graph (started w/ graph)

homotopy type Γ = wedge of circle.
(by squashing maximal tree)

b/c generators $\longleftrightarrow$ edges outside max. tree.

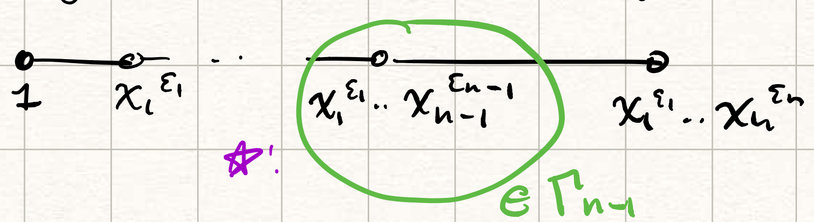
↖ tells you stuff like
$\operatorname{rank}(H) - 1 = [G:H](r(F)-1)$.

wedges of circles = basis X

($\Rightarrow$) Build the tree.
$\Gamma = \operatorname{Cay}(F, X)$. Prove that Γ free.

Γ_n = n-sphere around 1.
= reduced words in F of len n.
= $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$, $x_i \in X$.
where $x_i = x_{i+1} \Rightarrow \varepsilon_i \neq -\varepsilon_{i+1}$

in Γ, the word is represented by a geodesic from 1, of len n.



★!

$\Rightarrow (*)\ \Gamma_n \xrightarrow{\varphi_n} \Gamma_{n-1}$ inverse system

$(\Gamma_n), (\varphi_n)$

Pointed trees $\Longleftrightarrow$ Inverse System

vertex $\bigcup G_n$ $\longleftarrow$ (G_n, φ_n)

edges $(p, \varphi_n(p))$ Serre's trees

<u>Free product of $A * B$</u>

What do reduced words look like?

→ Pick non-trivial elements of A, B, and then concatenate as in free grp.

→ wew', $w, w' \in A * B$

$\overset{\shortparallel}{ww'}$ $e = id$ of $A * B$.

→ $w\,xy\,w' = wzw'$ $w, w' \in A * B$

$x, y, z \in A$

OR $x, y, z \in B$

Reduced words are all conjugate or can be inverted to the following:

$a_1 b_1 a_2 b_2 \cdots a_k b_k$.

Pf of Ping Pong

$H = \langle a, b \rangle \overset{?}{\cong} A * B$.

To show that a reduced word from $A * B$ is non-triv. in G by using action on X.

$w = a_1 b_1 \dots a_k b_k a$ (**)

$\Rightarrow wX_2 \subseteq a_1 b_1 \dots a_k b_k X_1$

$\subseteq X_1$

$a_i \neq 1$ b/c $a \in A \setminus \{1\}$

$b_i \neq 1$ "

Since $\exists p \in X_2 \setminus X_1$ that w takes to a differenciated pt of X, $w \neq 1$ in G.

**: was only one type of word.

others:

$b_1 a_1 b_2 a_2 \dots b_k$

Conjugate

$(b_1 a_1 b_2 b_2 \dots b_k a_k)^{-1}$

$= x a_k^{-1} b_k^{-1} \dots a_1^{-1} b_1^{-1} x^{-1}$

⇝ when you conjugate, you want

$x a_k^{-1} \neq id$

$x a_k^{-1} \in A$,

⇝ so if you work out details,

get $|A| \geq 3$.

$|B| \geq 2$. ← so NOT id.

example:

$G = SL(2, \mathbb{C})$

$M_1 = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $|z| \geq 2$

$M_2 = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$

exerc: $\langle M_1, M_2 \rangle \cong F_2$.

Sanov: $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$M_2 = J M_1 J$

Reduced words in M_1 & M_2 are non-zero in $SL(2, \mathbb{C})$ → WLOG can look at reduced words in M_1, J

$\langle M_1, J \rangle \cong \langle M_1 \rangle * \langle J \rangle$

Indeed, $X = \mathbb{C} \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az+b}{cz+d}$$

$X_1 = \{z \in X \mid |z| < 1\}$

$X_2 = \{z \in X \mid |z| > 1\}$

<u>Check</u>: $M_1^n X_1 \subset X_2$

$J X_2 \subset X_1$

Ping Pong for cyclic grps

Let G act on a set X.

Let x, y generate infinite cyclic subgrps

$\exists$ disjoint non-empty subsets of X,

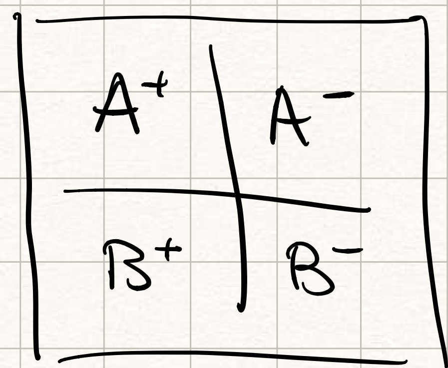
A^+, A^-, B^+, B^- st

$x(X \setminus A^-) \subset A^+$

$x^{-1}(X \setminus A^+) \subseteq A^-$

$y(X \setminus B^-) \subseteq B^+$

$y^{-1}(X \setminus B^+) \subseteq B^-$

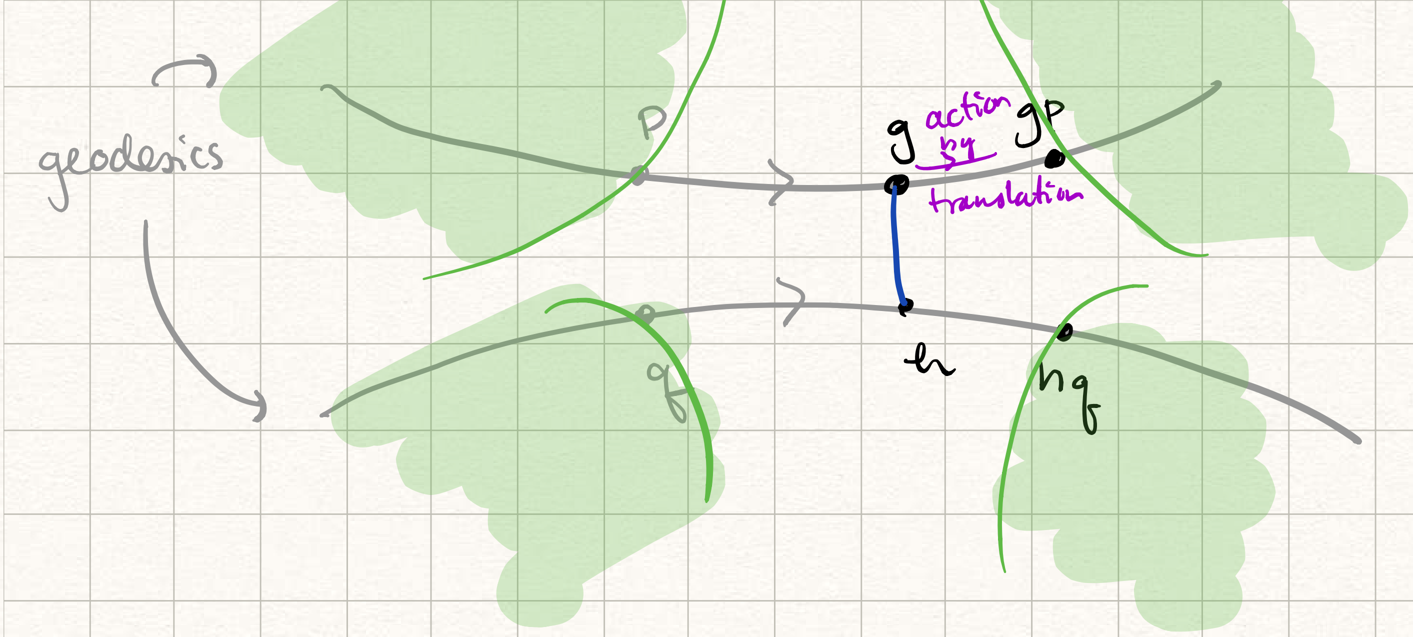


Then $\langle x, y \rangle \cong F_2$.

<u>exerc</u>: Deduce this ? ..

Grp G acting on a tree T

<u>Fact</u>: elements ∞-order act by translation on bi-infinite geodesic



hint: the 4 sections are A^+, A^-, B^+, B^- to play Ping Pong.

exerc: g, h hyperbolic, ℓ_g, ℓ_h trans. axis, disjoint.