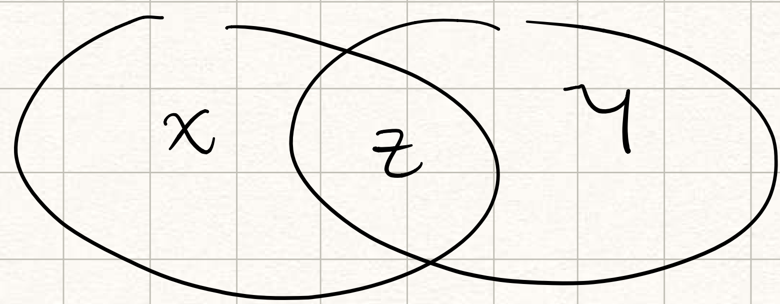
Bass - Serre Theory

Topological Origin

Seifert van Kampen

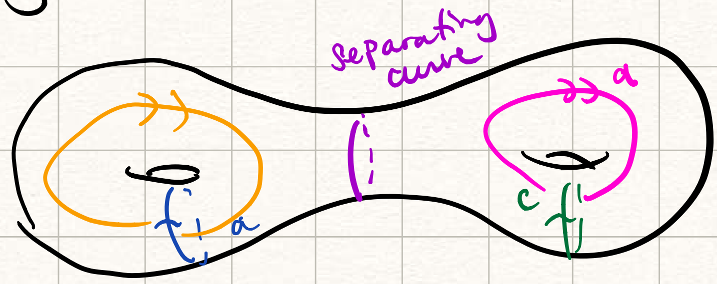


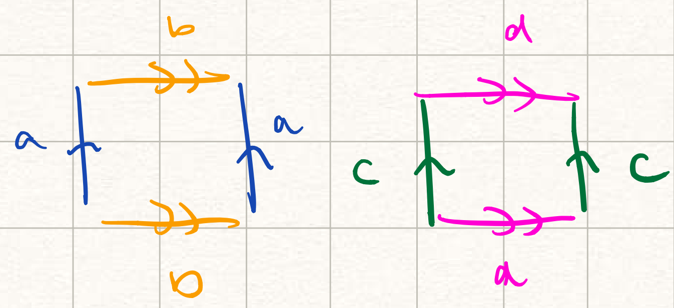
X, Y, Z path connected

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, hook] & X \\ Y & \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(Z) \arrow[r, hook] \arrow[d, hook] & \pi_1(X) \arrow[d] \\ \pi_1(Y) \arrow[r] & G \end{tikzcd}$$

Amalgams





Formally,

$$\begin{tikzcd} C \arrow[r, "i_B"] \arrow[d, "i_A"] & B \arrow[d, dashed] \\ A \arrow[r, dashed] & A \ast_C B \end{tikzcd}$$

largest grp, which is gen. by A, B

but $i_A(c) = i_B(c) \quad \forall c \in C.$

$$A \ast_C B = \langle A, B \mid i_A(c) = i_B(c) \ \forall c \in C \rangle$$

Reduced words

an element of $A *_C B$ not in C
can be written as:

$g_1 \dots g_k$ $\quad k \geq 1$, st
g_i come altern. from $A \setminus C$, $B \setminus C$.

<u>compare</u> such words:

$g_1 \dots g_k = h_1 \dots h_l$ ~~then~~ $k = l$

and $g_1 C = h_1 C$

$C g_k = C h_k$

$C g_i C = C h_i C \quad i \neq 1, k.$

↳ think of it in terms of A, B interactions.

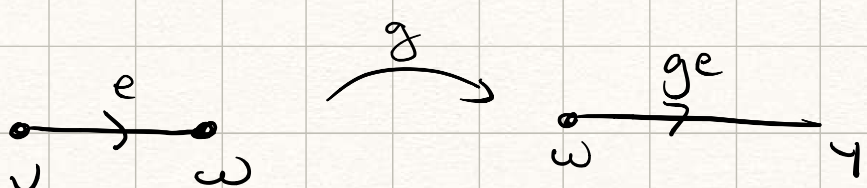
Build the BS tree for $G = A *_C B$

G acts on tree V(T) simplicially

$V(T) \to V(T)$

$E(T) \to E(T)$

without inversions (no edge flipping).



If g fixes a pt in an edge e,
then g fixes <u>all</u> of e.

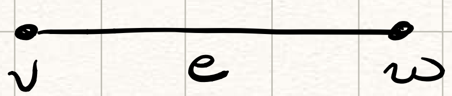
<u>Fundamental Domain</u> (fd)

G acting on a tree T,
Then fd of this action is a subgraph Y of T st $Y \cong G \backslash T$.

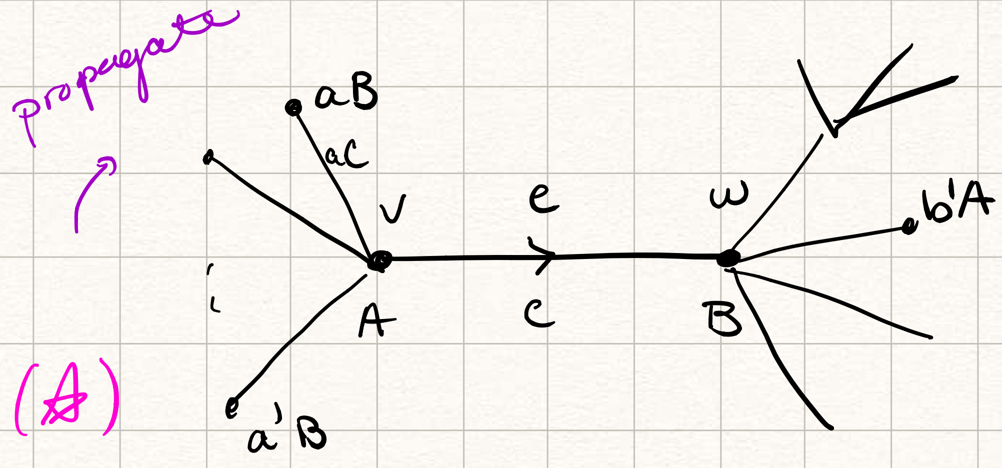
<u>Prop</u>: Fundamental domain exists
$\Leftrightarrow G \backslash T$ is again a tree.

$\pi_1(G \backslash T) = 1 \qquad \pi_1(G \backslash T) \neq 1$.

<u>Thm</u>: A grp G decomposes as an amalgam $A *_C B$
$\Leftrightarrow$ G acts on a tree (simplicially) w/ 2 orbits of vertices and 1 orbit of edges w/ fundamental domain



$G_v \cong A$, $G_w \cong B$, $G_e = C$.



$a \in A \setminus C$; edges move out but v gets fixed.

if $a \in C$, For rep. a from A/C.

vertices $\Longleftrightarrow$ $G/B \sqcup G/A$

edges $\Longleftrightarrow$ G/C

given by the map

$$\begin{tikzcd} G/C \arrow[r, "a"] \arrow[dr] & G/A \\ & G/B \end{tikzcd}$$

Q: Why is this a tree?

Graph (V, E)

Why is this a tree?

Use inverse system lemma

$$W_n = \left\{ \begin{array}{l} \text{reduced words w/} \\ \text{length } n \end{array} \right\}$$

inverse sys: $T_n \to T_{n-1}$

pt: coset of A

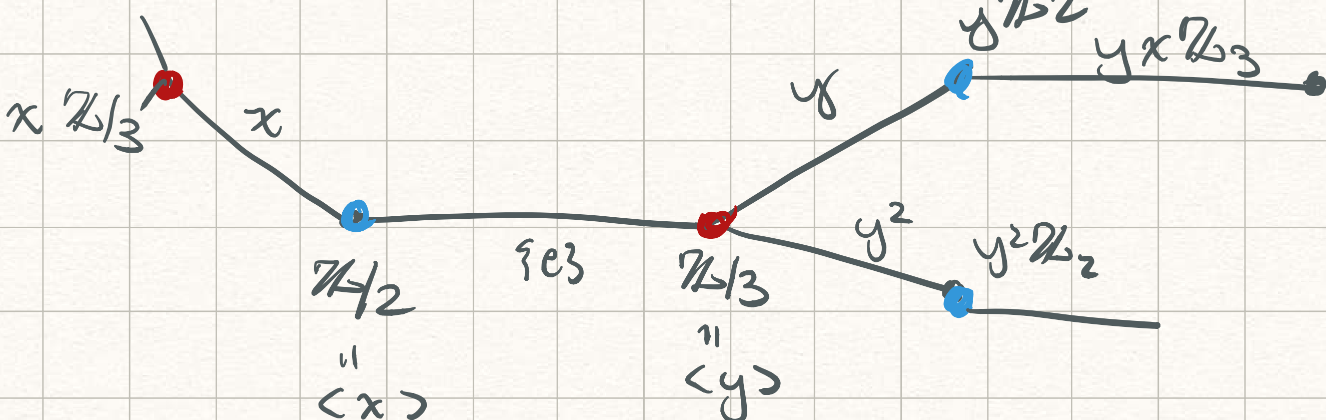
$$g_1 \cdots \underset{\substack{\rotatebox{90}{$\in$} \\ B \setminus C}}{\underline{g_k}} A$$

$$g_1 \cdots \underset{\substack{\rotatebox{90}{$\in$} \\ A \setminus C}}{g_k} B$$

<u>explicit ex</u>:

$$PSL(2,\mathbb{Z}) = \langle x, y \mid x^2 = y^3 = 1 \rangle$$

$$G_v \cong A = \mathbb{Z}/2$$
$$G_w \cong B = \mathbb{Z}/3$$
$$G_e \cong C = 1$$



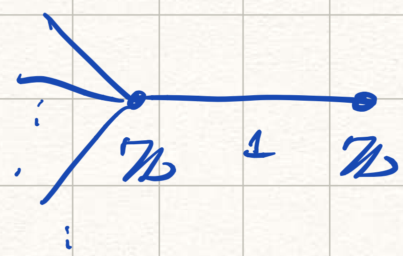
me: the acyclicity comes from the reduced words.

if you cycled, then

Q: Is BS q-isom to Cay(G)?

A: NOT:

$$F_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$$



∞-valence!

Stallings

If $G \curvearrowright T$ co-compactly w/ finite stabs, then G is virtually free (finite rank).

Q: Role of C?

A: Valence $= A/C,\ B/C$.

☆: $1 \xrightarrow{\text{gend}}$ vert in BS tree $=$ a reduced word in the amalgam sense.

Acyclicity comes from inverse sys (if you had loop, $\Gamma_n \to \Gamma_{n-1}$ not well-defined)

OR the fact that would induce cancellation in inverse sys, which is a contradiction.