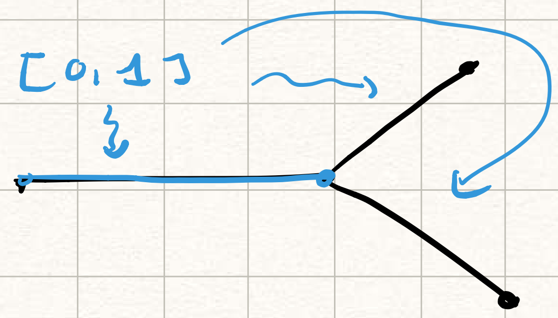
CAT(0) Cube Complexes

Motivation: Trees = nicest object possible.

(here we assume simplicial trees).

ie



- Trees are CAT(0).
- Trees are 0-hyperbolic
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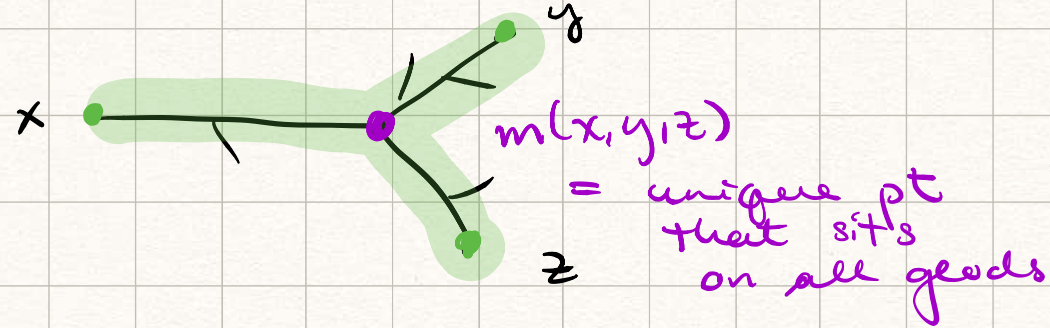
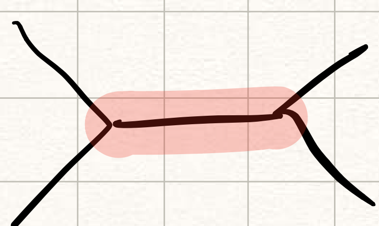


- every edge separates trees into two components.



Def: $V = A \cup A^c$

$V = B \cup B^c$

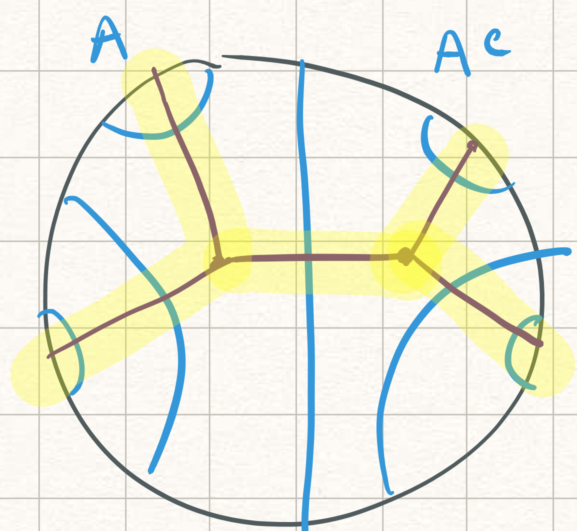
A, B are nested if one of these is empty:

$A \cap B$

$A \cap B^c$

$A^c \cap B$

$A^c \cap B^c$



POINT: trees connect nested components.

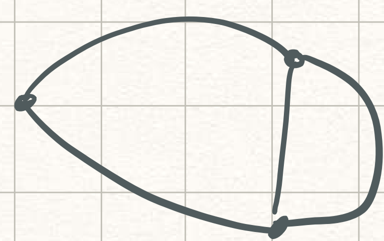
- Action on trees (Bass-Serre theory) $\Rightarrow$ Alg info on grp.

CAT(0) Cube Complex

Def: cube complex

= complex obtained by gluing unit Euclidean cubes $[0,1]^n$ by isometrically id-ing their faces.

ex: • graphs are 1D cube complexes. $(n = 1)$.



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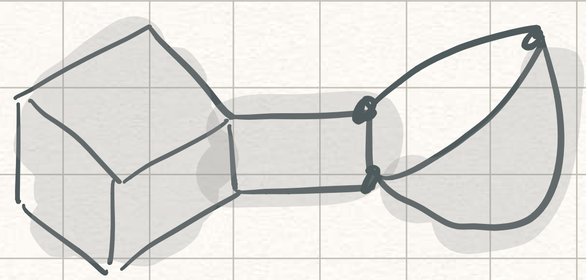

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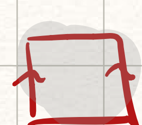
Note: this

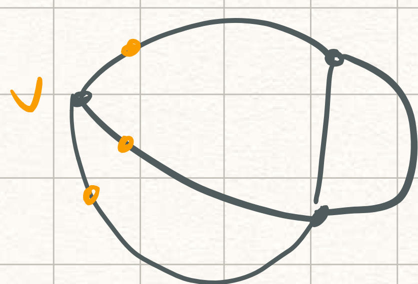


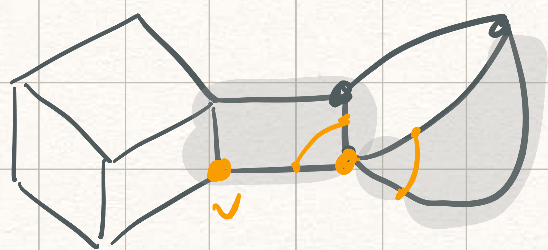
is allowed, but NOT simplicial.

Def: Link of a vertex in X,

$Lk(v, X)$ = "a vertex whose vertices are the edges at v"

a "simplex where edge for two vertices which span a square"



- 

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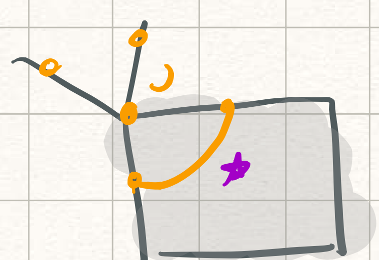
- 

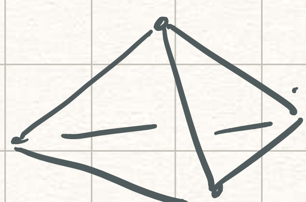
Def: A cube complex is locally CAT(0) at v if $Lk(v, X)$ is a flag simplicial complex.

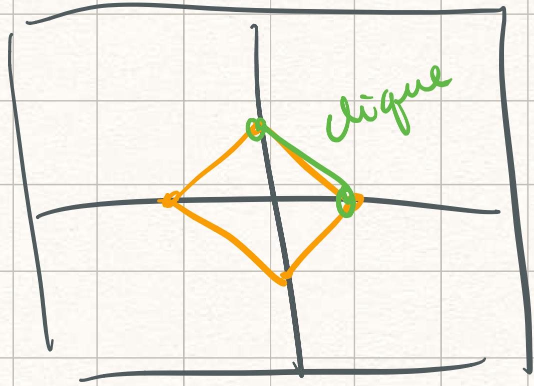
Def: simplicial = simplices intersect at face.

NOT simplicial intersection = • •

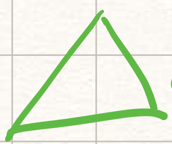
each • = face,
but both ≠ face.

Def: flag = every clique (in the 1-skeleton) spans a simplex.

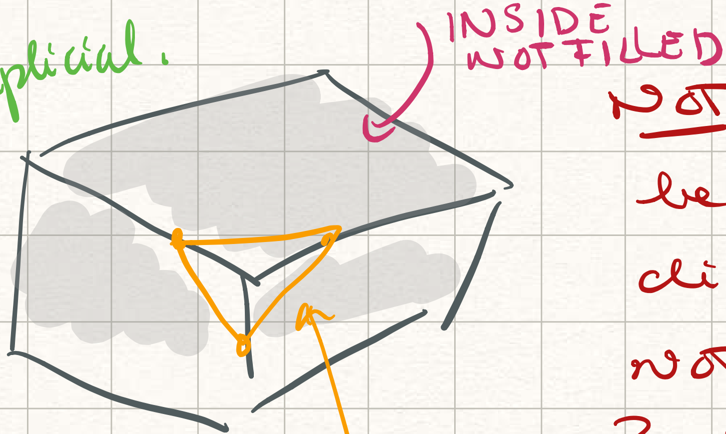




simplicial & flag



is simplicial.



NOT flag because Δ clique does not span 2-simplex

so this Δ isn't filled

<u>Def</u>: A cube complex X is CAT(0) ⟺ it is simply connected & locally CAT(0) at every vertex (⟺ all links are flag simplicial).

<u>Rmk</u>: A cube complex is NPC (non-positively curved) if it is a each vertex.

⇒ Universal cover of NPC cube complex is CAT(0).

$\overset{\text{CAT(0)}}{\text{2 + CC.}}$ $\overset{\text{CCC}}{''}$

<u>ex</u>: · Trees are CAT(0) cube cmplx

· Products of CCC are CCC.

· <u>Salvetti complexes</u>

= finite simplex graph Γ

Torus $\rightarrow$ $\mathbb{T} = \prod_{V(\Gamma)} S^1$

for every clique $J \subseteq V(\Gamma)$

$$\mathbb{T}_J = \prod_{v \in J} S^1 \times \prod_{v \notin J} \bullet$$

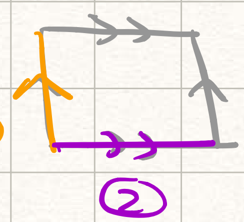
$$S_\Gamma = \bigcup_{J \text{ clique of } \Gamma} \mathbb{T}_J \subseteq \mathbb{T}$$

$\leftarrow$ gluing made by inclusion.

S_Γ · is NPC

Salvetti

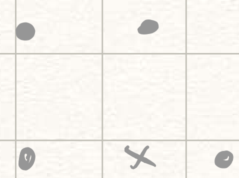
<u>ex</u>: i) $\Gamma = \overset{1}{\bullet}\!-\!\!\!-\!\!\!-\overset{2}{\bullet}$

$\mathbb{T} =$ 

$\mathbb{T}_{\{1\}} = \bigcirc \times \circ$

$\mathbb{T}_{\{2\}} = \bigcirc \times \cdot$

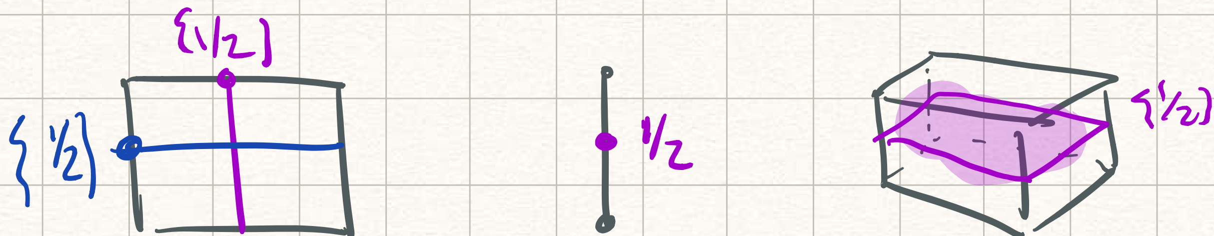
$\Rightarrow T_{\{1,2\}} =$ 

ii) $\Gamma =$ 

$S_\Gamma =$ 

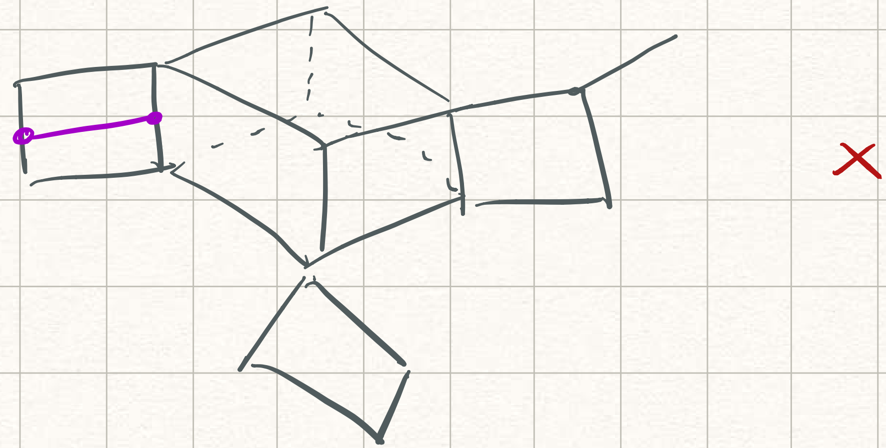
Hyperplanes & Half-Spaces

Def: A midcube of a cube $[0,1]^n$ is the set $\{\frac{1}{2}\} \times [0,1]^{n-1}$

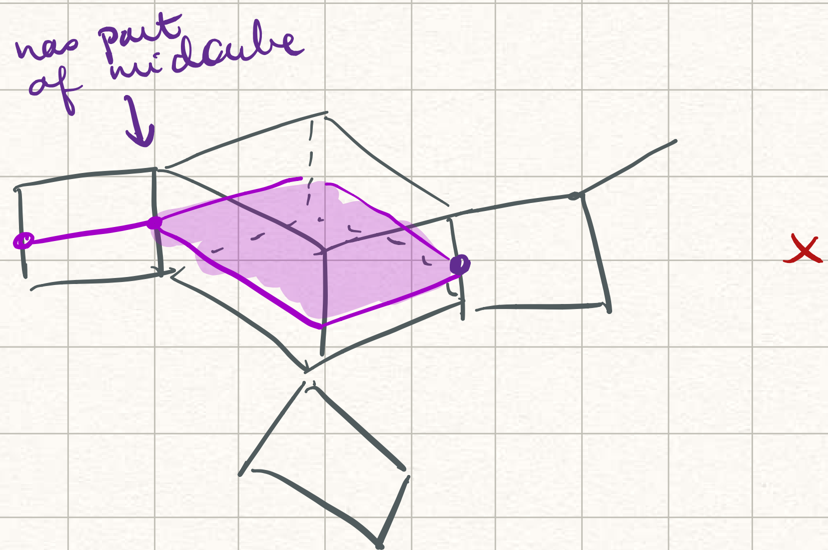


Def: A hyperplane in a cube complex is a collection of midcubes which is non-empty minimal (wrt inclusion) st it is <u>closed</u> under inclusion of midcubes.

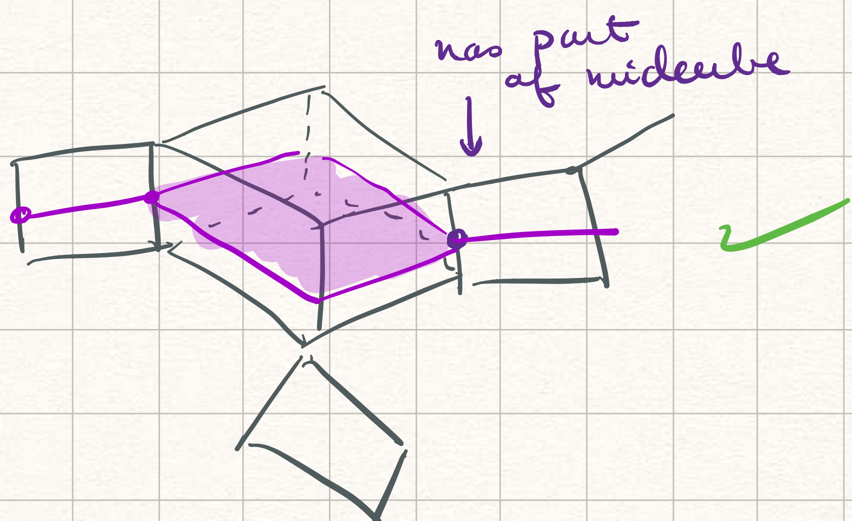
1)



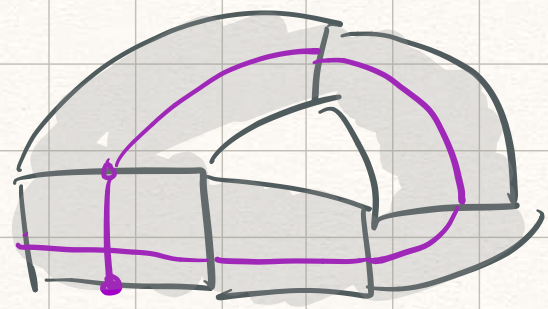
2)



3)



Now we've chosen <u>ONE</u> hyperplane!



we ended up crossing our hyperplane.

→ this kind of pathology does NOT happen in CAT(0) cube complexes!

⇝ in CCC, hyperplanes will section CCC in half, just like edges in trees!

Q : what's special about cubes?
A : Behave well in ℓ^1-metric.