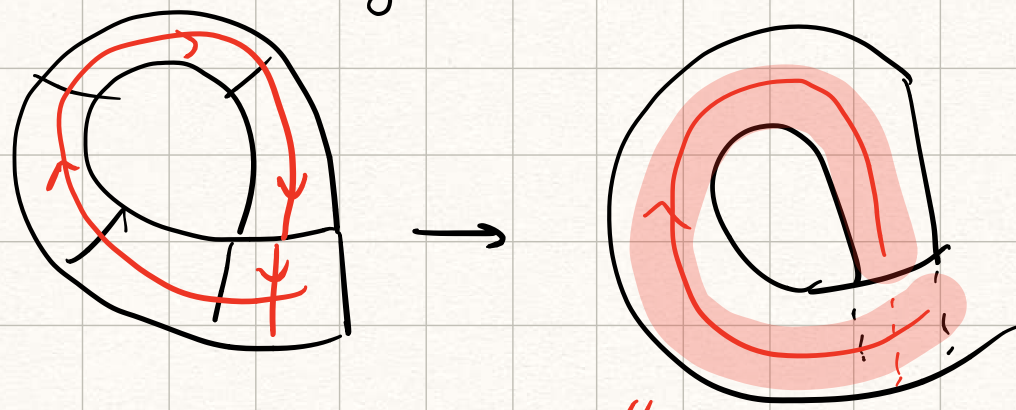
Prop: if X is CCC and h^n is a hyperplane, then

part I
- h^n is non-self intersecting (embedded)
- convex subset
- CAT(0) cube complex (CCC)
- have a product nbhd
  $N(h^n) = h^n \times I$ interval

part II
- $X - h^n$ has 2 components

Pf (I): Build an "abstract copy of $N_{1/2}(h^n)$ remembering an orientation"



N''

this has 2 layers now so NOT intersect.

check: get CCC.

N is a loc. CAT(0) cube cmplx.

= NPC

$\tilde{N} \longrightarrow N \longrightarrow X$

univ cover = CCC

$\tilde{N} \to X$ is a local isom.

(why) ★ $\Longrightarrow$ CAT(0) geometry $\tilde{N} \to X$ is an isometry

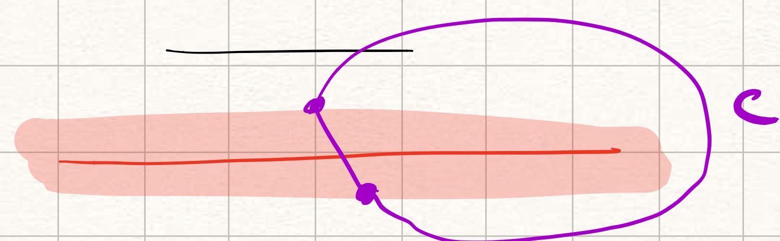
★: local to global phenomenon. (see in Conlan's talk)

$\tilde{N} = h^n \times [0,1]$

X connected, h^n connected so prod nbhd.

(II) # comp ≤ 2 because can at most cross N.

Assume < 2. Then look at c



Define map :

$X \to \underbrace{[0,1] / 0 \sim 1}_{\text{wrap } N \text{ onto itslf}} = S^1$

and everything else to 0.

$x \notin N \mapsto 0$ map outside N to a pt

$x \in N \mapsto t$

$\overset{\shortparallel}{(x', t)}$

$\cap$

h^n

we have one comp

$\to$ loop $\underset{\shortparallel}{c}$ in X

p

which maps to gen. of $\pi_1(S^1)$

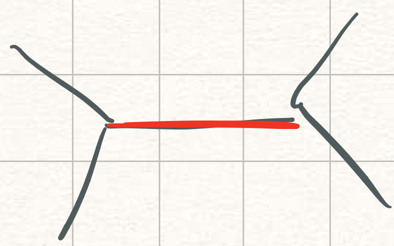
$\Rightarrow$ contradiction of 1-connect. of X. $\square$

II: Not one comp $\rightsquigarrow$ so 2.

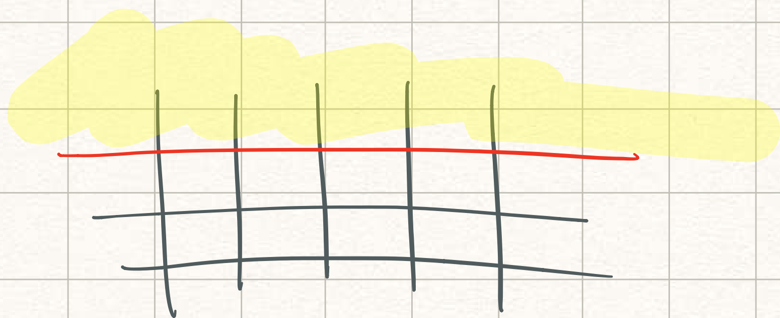
$N(h^n)$ = carrier of h

$X \setminus N(h^n)$ has 2 comps which are all halfspaces.

ex:



half-spaces = half trees.



If h is half-space,
h' boundary hyperplane
h^c "other" half sp.

Metrics on CCC

1. CAT(0) metric d_2
2. Combinatorial metric d_1

On the vertices d_1 = path metric of 1 skeleton.

Prop (the distance formula)

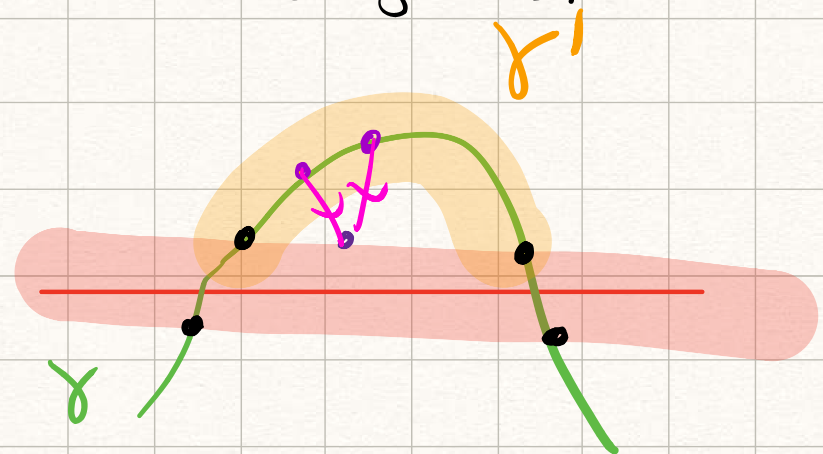
$v, w \in X^{(0)}$

$$d_1(v,w) = \# \left\{ \begin{array}{l} h \text{ half-sp} \\ \text{st } v \in h \\ w \notin h \end{array} \text{ and} \right\}$$

$= \#$ hyperplanes sep v, w.

Pf: $\geq$ ✓

$\leq$ a d_1-"geodesic" crosses a hyp at most once (otw not a geod).



- project γ' to $N(h'')$, $h' \times \{1\}$

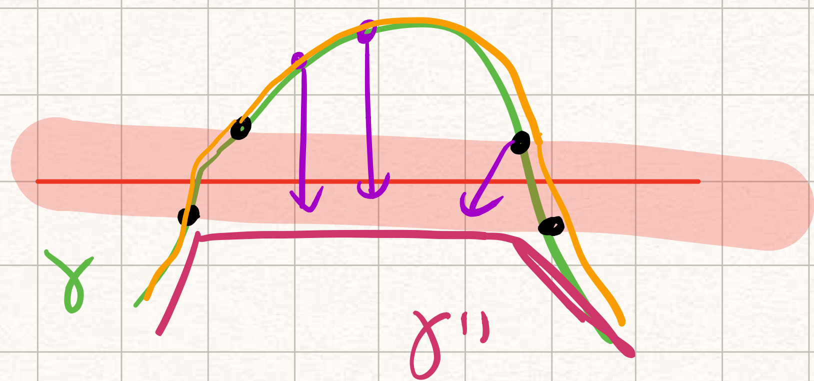
exc: vertices of CCC project to vertices of $N(h'')$.

hint: otw the proj is in $\mathrm{Int}(N)$.

$C \subseteq X$ convex
CAT(0)

$\pi : X \to C$
$x \mapsto \pi(x)$ closest to x in C

π : 1-Lip $\Rightarrow$ $\pi(\gamma')$ is again a path in 1-skeleton save edges
can only contract (CAT(0) prop)
$\Rightarrow \gamma''$ is shorter than γ' contra.

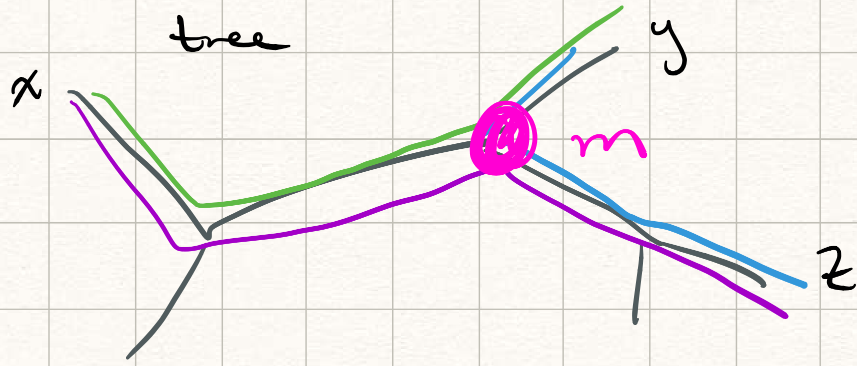


Median sp (X, d)

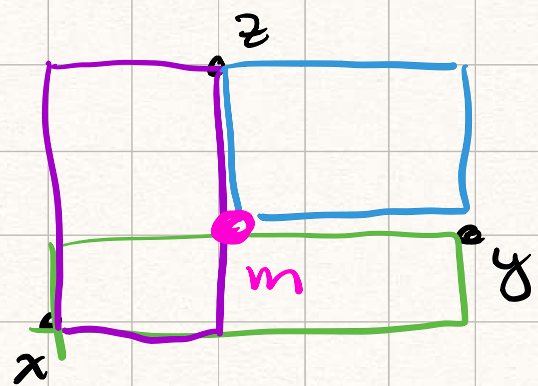
$I(x,y) = \{ z \in X \mid d(x,z) + d(y,z) = d(x,y) \}$

(X,d) is median if $\forall x,y,z$, $\exists! \, m = m(x,y,z)$
st $\{m\} = I(x,y) \cap I(y,z) \cap I(x,z)$

ex!



ℓ^1



Prop X is CCC

$\Rightarrow X^{(1)}$ is a median graph

& $X^{(1)}$ uniquely id's CCC.