Construction:

1) $K := \underset{t}{\mathbb{Z}} * (\underset{x}{\mathbb{Z}} \times \underset{y}{\mathbb{Z}})$ via fp $\langle x, y, t \mid [x,y]=e \rangle$

2) $\forall (r,s) \in \mathbb{Z}^2$, define $t(r,s) := y^s x^r t x^{-r} y^{-s}$

5) For $M > a \geq 0$, $N > b \geq 0$

define

$$K^{M,N}_{a,b} := \langle t(a,b), x^M, y^N \rangle \quad (\leq K)$$

7) <u>Lem</u>: $K^{M,N}_{a,b} \cong K$ via

$$t(a,b) \mapsto t$$
$$x^M \mapsto x$$
$$y^N \mapsto y$$

9) Let $M = \{(a_i, b_i, c_i, R) \mid i \in I\} \cup$
$\{(a_j, b_j, c_j, L) \mid j \in J\}$

be a MM w/ modulus m in which $(0,0)$ is terminal.

10) Define $\forall i \in I$, the map

$$\begin{tikzcd} K_{a_i,b_i} \arrow[rr, "\phi_i"] \arrow[dr, dashed] & & K^{m^2}_{c_i,0} \\ & K \arrow[ur, dashed] & \end{tikzcd}$$

via

$$t(a_i,b_i) \mapsto t(c_i,0)$$
$$x^m \mapsto x^{m^2}$$
$$y^m \mapsto y$$

simulate $\binom{L}{R}$ step of MM

11) Define $\forall j \in J$, map

$$\psi_j : K^{m,\,m}_{a_j, b_j} \longrightarrow K^{1,\,m^2}_{a_j, b_j} \quad \text{via}$$

$$t(a_j, b_j) \longmapsto t(0, c_j)$$

$$x^m \longmapsto x$$

$$y^m \longmapsto y^{m^2}$$

↑ simulate $\binom{R?}{L}$ step of MM

12) Lem: ϕ_i, ψ_j isom $\forall i, j$.

13) Define HNN ext

(w/ stable letters $\{v_i \mid i \in I\}$
$\{\ell_j \mid j \in J\}$

via $K_M = K *_{\{\phi_i\}_{i \in I} \cup \{\psi_j\}_{j \in J}}$

14) Define $\langle t \rangle' = \langle t, v_i\ \forall i \in I, \ell_j\ \forall j \in J \rangle$
$(\leq K_M)$

↑ conjugation by v_i or ℓ_j is 1-step in MM

23) Lem: $t(\alpha, \beta) \in \langle t \rangle$
$\iff (\alpha, \beta) \in H_0(M)$

$t(0,0) = t$

$(\alpha, \beta) \xrightarrow[M]{} (0,0)$

$t(\alpha, \beta)$.

Cred: Andre (a)° & Cohen.

(Idea: I thought F_2 wasn't complex,
but I also thought TM weren't
& they are everything...
it turns out can't judge complexity
by how hard it is to understand
struct.)

- So far you can "decide membership"
  by conjugating → the number change
  is the MM. Want WP.

21) Define HNN ext, $f := \mathrm{Id} : \langle t \rangle \to \langle t \rangle$

$$G_M := K_M *_f$$
$$= \langle K_M, k \mid k h k^{-1} = h \quad \forall h \in \langle t \rangle \rangle$$

(NB: G_M, K_M both fp)

26) Lem: $k\, t(\alpha, \beta)\, k^{-1} = t(\alpha, \beta)$ in G_M.

$\iff t(\alpha, \beta) \in \langle t \rangle$

$\iff (\alpha, \beta) \in H_0(M)$

(10.4) $\Rightarrow$ So G_M is fp grp w/
undecidable WP.

Pf: Take TM T w/ $\Omega(T) = \mathbb{K}$

- Construct M from T, then G_M from M.

So can solve WP in G_M

$\iff$ Can decide membership in $\mathbb{K}$.

(Turing res)

Thm (12.1)

There exists an (algebraic) construct that on input of fp,

$P = \langle X \mid R \rangle$, & words $w \in X^*$,

outputs a new fp $P(w)$ &

hom $\phi : \overline{P} \to \overline{P(w)}$ st

1) if $\overline{w} \neq e$ in $\overline{P}$, then ϕ is an embedding, hence $\overline{P(w)} \neq \{e\}$

2) if $\overline{w} = e$ in $\overline{P}$ then $\overline{P(w)} = \{e\}$.

Pf: Take $P = \langle X \mid R \rangle = \langle x_1, \dots, x_n \mid R \rangle$

→ form free prod. $\overline{P} * F_2$ w/ pres:

$P * \langle a, b \mid - \rangle$

→ take another copy of F_2 w/ pres

$S = \langle c, d \mid - \rangle$.

Form $A \le \bar{P} * F_2$ by

$A = \langle b, aba^{-1}, a^2bab^{-1}a^{-2}, a^3(\bar{w},b)a^{-3},$
$a^{(3+i)} x_i a^{-(3+i)} \quad \forall x_i \in X \rangle$

Will form amalgamated prod!

If $\bar{w} \ne e$ in $\bar{P}$, then A is free on this gen set (normal forms).

This $A \cong F_{n+4}$.

Form $B \le F_2$ via

$B := \langle d, cdcd^{-1}, c^2dcd^{-1}c^{-2}, c^3dc^{-3},$
$c^{(3+i)} d c^{-(3+i)} \quad \forall x_i \in X \rangle$

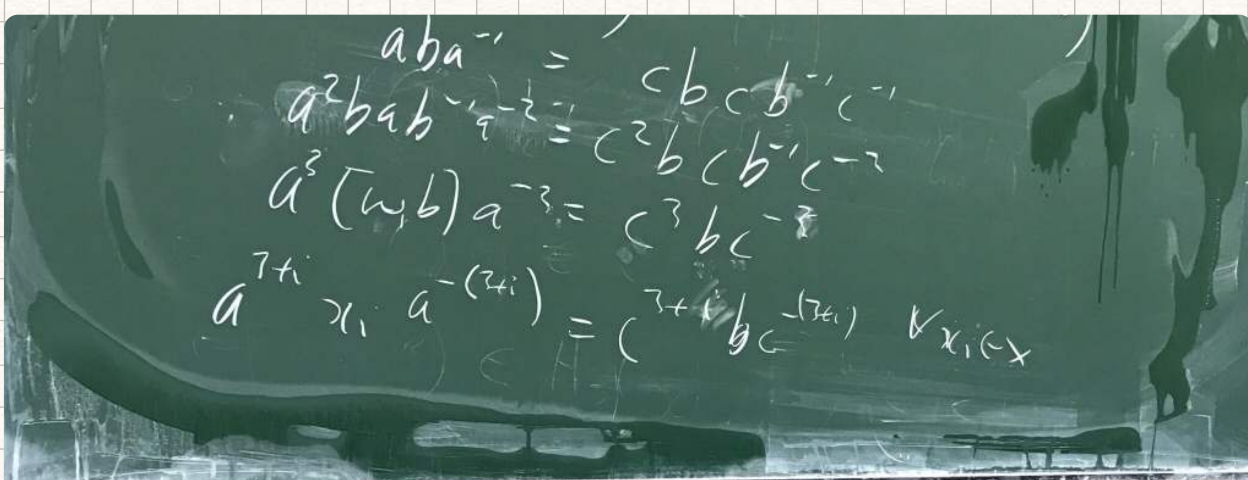
B is free on this gen set ($\Rightarrow B \cong F_{n+4}$).

Let $\varphi : A \to B$ be the map which gives the natural isom when $\bar{w} \ne e$ in $\bar{P}$.

→ Let $p(w)$ be the fp given by

GENS: $X \cup \{a, b, c\}$

RELS: $R \cup \{ \cancel{b = d} \}$ replace a w/ b.



If $\overline{w} \neq e$, then $p(w)$ is fp of the amal. prod. $(\overline{p} * F_2) *_{\varphi} F_2$ so $\overline{p} \hookrightarrow \overline{p(w)}$.

If $\overline{w} = e$ in $\overline{p}$ then (check) $\overline{p(w)} \cong \{e\}$

<u>Conj</u>: (12.3)

A alg that on input fp p decides if $\overline{p} \cong \{e\}$.

<u>Pf</u>: Take fp Q (WP

- Take w o Q
- Form $Q(w)$
- Then $\overline{w} = e$ in $\overline{Q} \iff \overline{Q(w)} = \{e\}$.

<u>Def</u>: A property ρ of grp is a Markov property if

$\exists$ fp G_+, G_- st

1) G_+ has ρ

2) G_- not e & if $G \overset{fp}{\hookrightarrow} H$ then H not e.

Thm: Let ρ be a Markov prop.

Then $\nexists$ alg that on input of fp p, computes if $\bar{p}$ has ρ.

Pf: Let ρ be Markov, P_+, P_- fps of G_+, G_-.

- Let Q be fp ct of grp w/ IWP
- take $w \in Q$ from fp $(P_+) * ((P_- * Q)(w)) = S$
- If $\bar{w} = e$ in Q, then $\bar{w} = e$ in $\bar{P}_- * \bar{Q}$

  so $\overline{(P_- * Q)} = \{e\} \Rightarrow \bar{S} \cong \bar{P}_+$ has ρ

- If $\bar{w} \neq e$ in $\bar{Q} \Rightarrow \overline{P_-} \hookrightarrow \overline{P_+} * \bar{Q}$

  $\hookrightarrow \overline{(P_- * Q)(w)} \hookrightarrow \bar{S}$ so $\bar{S}$ not ρ.

Thus $\bar{w} = e$ in $\bar{Q} \Leftrightarrow \bar{S}$ has ρ.

Markov: free, finite, Ab, cyclic, solvable, WP, etc.