Groups of Homeo(S^1)

PLAN:

I. Fundamentals

II. Tits alternative
for $\mathrm{Homeo}^+(S^1)$.

(Tits in gen: fg $H \leq GL_n$,
then H either is free
or solvable.)

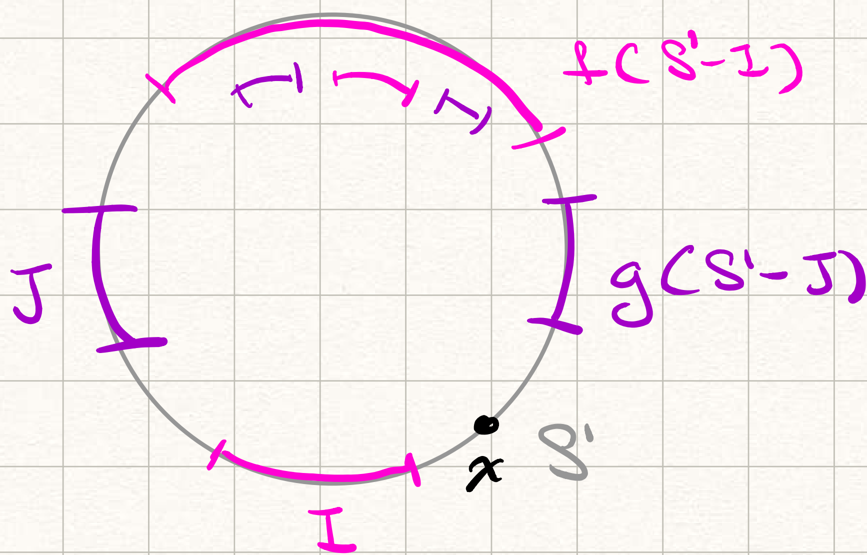
Any subgrps $\Gamma \subset \mathrm{Homeo}(S^1)$
either contain F_2 or there
is Γ-invariant probability
measure on S^1.

III. A theorem about
$\mathrm{Aut}(\pi_1 \Sigma_g) \curvearrowright \partial \pi_1 \Sigma_g = S^1$,
$g \geq 2$.

<u>examples</u>:

- $\mathrm{Aut}(\pi_1 \Sigma_g)$
- $SO(2)$
- $PSL_2(\mathbb{R}) \curvearrowright \mathbb{RP}^1 = S^1$
  by Mobius trans.
- Thompson's F acts on $[0,1]$,
  $F \subset T$ hence by id'ing $0=1$,
  $T = \langle F, x \mapsto x + \frac{1}{2} \bmod 1 \rangle$.

- F_2 <u>example</u>: $F_2 = \langle f, g \rangle$

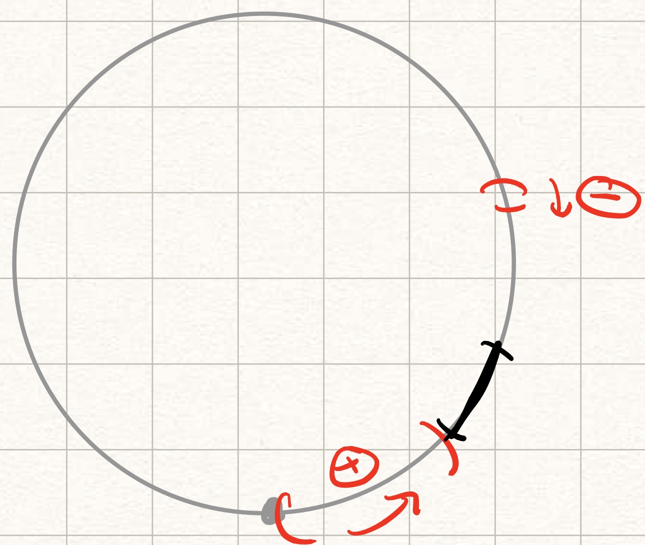


<u>Claim</u>: $w \in F_2$
$\mapsto w(f,g)(x)$
is injective.

(Pf (Ping Pong)
Induct on word len. exc.)

- $\mathbb{Z} = \langle f \rangle$ where f specified by:
    - K closed set $= \mathrm{fix}(f)$
    - On each complimentary interval $\simeq \mathbb{R}$, f look like $x \mapsto x+1$.

Assignment



Fact: $(K, \pi_0(S^1 - K) \to \{+,-\})$ is a complete conjugacy invariant on $\mathrm{Homeo}^+ S^1$ iff $K \neq \emptyset$.

Rmk: Not generally. Don't know what to do w/ rotations.

Poincaré Rotation

Def:

$$\begin{array}{c} \mathbb{Z} = \langle x \mapsto x+1 \rangle \\ \downarrow \\ \mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}} \\ \downarrow \\ \mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z}) \end{array}$$

$\mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$ = homeo of $\mathbb{R}$ commuting w/ $x \mapsto x+1$

translation #

- For $\tilde{f} \in \mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$, define $\widetilde{\mathrm{rot}}(\tilde{f}) = \lim_{n\to\infty} \left(\frac{\tilde{f}^n(0)}{n} \right) \in \mathbb{R}$.
- For $f \in \mathrm{Homeo}^+(S^1)$, $\mathrm{rot}(f) = \lim_{n\to\infty} \left(\frac{\tilde{f}^n(0)}{n} \right) \bmod \mathbb{Z}$.

Properties:

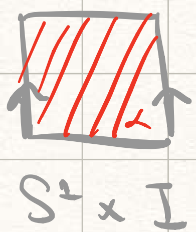
- $\widetilde{\mathrm{rot}}(x \mapsto x+\alpha) = \alpha$
- homogeneity:
  $\widetilde{\mathrm{rot}}(\tilde{f}^k) = k\, \widetilde{\mathrm{rot}}(\tilde{f})$

- Homomorphism (to $\mathbb{R}^+$) when restricted to abelian subgrps.

- Conjugacy invariant.

- Quasi-morphism :
$$\widetilde{rot}(\tilde{f}\cdot\tilde{g}) - \widetilde{rot}(\tilde{f}) - \widetilde{rot}(g) \leq 1.$$

<u>Minsky</u>:



glue top & bottom of I by $x \mapsto x+\alpha$, $\alpha \notin \mathbb{Q}$

$\sim$ 

← irrational slope α

anyway, get foliation by $///\alpha$

if instead, glue by f w/ $rot(f) = \alpha$ but <u>not</u> conjugate to $x \mapsto x+\alpha$.

Thm (Dynamical trichotomy)

$\Gamma \subset \mathrm{Homeo}^+(S^1)$. Either:

① Finite orbit.

② All orbits are dense.

③ There is a Γ-invariant closed subset K contained in the closure of every orbit, where the action on K is minimal.

K in ③ is homeo to Cantor set, and is unique.

Pf: Set of orbit closures $\{\overline{\Gamma x} \mid x \in S^1\}$ is partially ordered (inclusion).

Take a minimal elmt.

Finite $\Leftrightarrow$ ①

S^1 $\Leftrightarrow$ ②

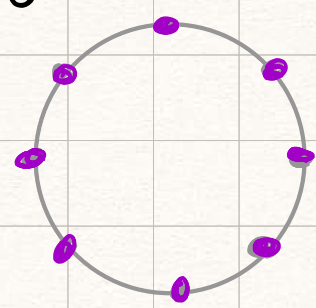
Else, Γ-invariant closed.

Every pt is an accumulation pt.

Empty interior. re-listen.

Pf of Tits Alternative

<u>Rk</u>: If Γ has finite orbit, you can put pt-masses there to produce inv. meas.

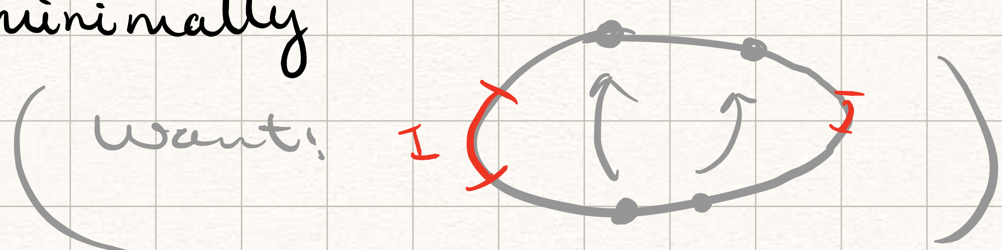


- If action is equi-continuous, then Arzela-Ascoli $\Rightarrow$ Γ is rel. compact. (so? Re-listen).

<u>ex</u>: Γ is conjugate to subgrp of $SO(2)$.

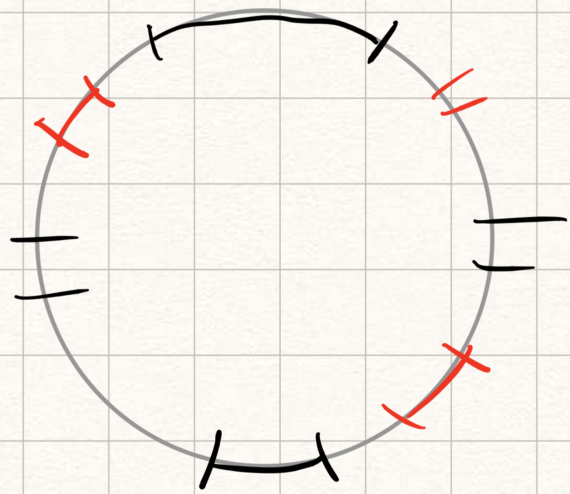
- If not, $\exists\, \varepsilon > 0$, $\exists\, (I_n)$, $I_n \to 0$, $\exists\, (g_n) \in \Gamma$, st $\mathrm{len}(g_n(I_n)) > 0$.

Assume (for a bit) that Γ acts minimally

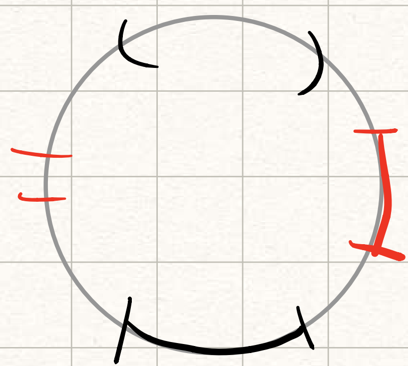


Problem (say in $SL_2(\mathbb{R})$)

① domains not connected



② then take as cover



nice picture $\Rightarrow$ Ping Pong.

(pass to subsequence)

$\Rightarrow \exists$ interval J of length $> \varepsilon/2$ w/ length of $g_n^{-1}(J) \to 0$.

minimal $\Rightarrow$ $\forall x \in S^1$ $\exists y$

st $\exists (h_n)$ w/ len$(h_n[x,y]) \to 0$.

<u>Define</u>:

$$\phi : S^1 \to S^1$$
$$x \mapsto \sup \left\{ y \mid [x,y] \text{ has this property} \right\}$$

<u>Fact</u>: ϕ is a finite order homeo which commutes w/ Γ.

So passing to a quotient, $\phi = id$, in which case you can play Ping-Pong.

To finish the pf (re-listen), conclude that assuming minimal action is no big deal.