Reference: E. Ghys
"Grps acting on the circle".

Picture: what does this look like?

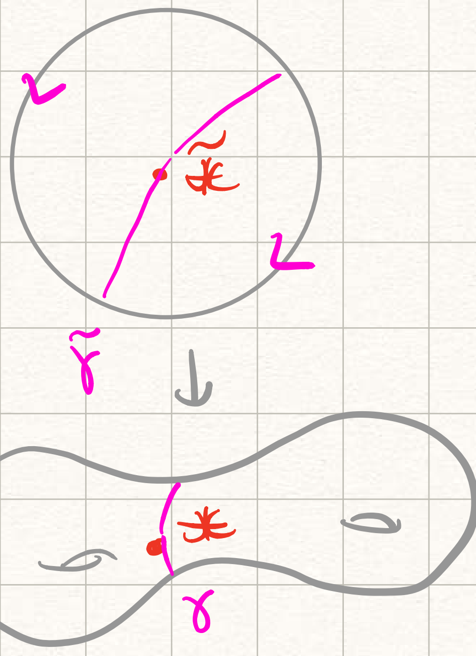
relisten. $\mathrm{Aut}(\pi_1 \Sigma_g) \curvearrowright \partial \pi_1 \Sigma_g = S^1$

$$\begin{tikzcd}
\mathrm{Inn}(\pi_1 \Sigma) \arrow[r, hook] & \mathrm{Aut}^+(\pi_1 \Sigma_g, *) \arrow[r] & \mathrm{Out}(\pi_1 \Sigma_g) \\
\pi_1 \Sigma_g \arrow[u, phantom, "\cong" rotate=90] \arrow[r, "\text{``push } \gamma\text{''}"'] & \mathrm{MCG}(\Sigma_g, *) \arrow[u, "\cong"'] \arrow[r] & \mathrm{MCG}(\Sigma_g). \arrow[u, phantom, "\cong" rotate=90]
\end{tikzcd}$$

deck transformation γ
$PSL_2(\mathbb{R})$

Alternate description of action
$\curvearrowright \mathrm{Aut}^+ \pi_1 \Sigma_g$: put hyperbolic struct on Σ_g:

$\mathbb{H}^2 = \tilde{\Sigma}_g =$



if $f(*) = *$

$\exists!$ lift $\tilde{f}$ fixing $\tilde{x}$, inducing a homeo of

$$\partial \mathbb{H}^2 = S^1,$$

depending only on $[f] \in MCG(\Sigma_g, *)$.

This gives

$$MCG(\Sigma_g, *) \to \mathrm{Homeo}^+(\partial \mathbb{H}^2)$$

Same ~~as~~ action of $\mathrm{Aut}^+(\pi_1 \Sigma_g)$.

Q: Suppose you have injective

$$\rho : \mathrm{Aut}^+(\pi_1 \Sigma_g) \to \mathrm{Homeo}^+(S^1)$$

Is ρ semi-conj. to the boundary action?

A : (Speaker)

Any $\rho: \mathrm{Aut}^+(\pi_1 \Sigma_g) \to \mathrm{Homeo}^+(S^1)$ is either semi-conjugate to the boundary action or is trivial. ($g \geq 3$).

Cor $g=2 \Rightarrow$ can factor through $\mathbb{Z}/10\mathbb{Z}$).

The rest will be the pf of this.

Pf: Suppose (assume $g \geq 3$) that $\rho: \mathrm{Aut}(\pi_1 \Sigma_g) \to \mathrm{Homeo}^+(S^1)$.

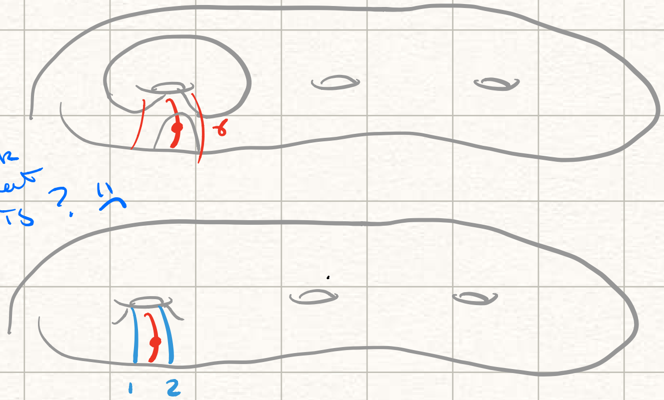
First study $\underline{\rho}|_{\pi_1 \Sigma_g}$.

the deck transfo from before.

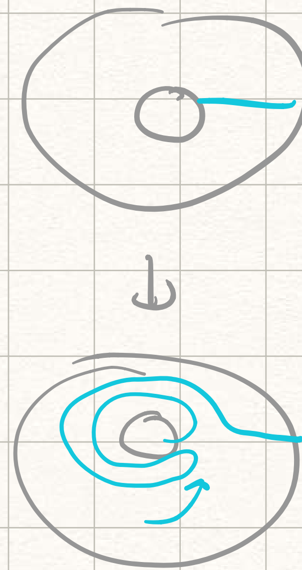
Lem 1: If $\gamma \in \pi_1 \Sigma_g$ represents a nonseparating simple closed curve.

then $\mathrm{rot}(\rho(\gamma)) = 0$.

IDK what γ is ? :'(



τ_1 = Dehn twist around 1



Pf lem 1: γ in MCG $= \tau_1 \tau_2^{-1}$

where τ_1 conj. τ_2

& τ_1, τ_2 commute.

$$\rho(\gamma) = \rho(\tau_1) \cdot \rho(\tau_2)$$

$$\to \text{rot}\, \rho(\gamma) = \text{rot}(\rho(\tau_1)\rho(\tau_2))$$
$$= \text{rot}(\rho(\tau_1)) - \text{rot}(\rho(\tau_2))$$
$$= 0$$

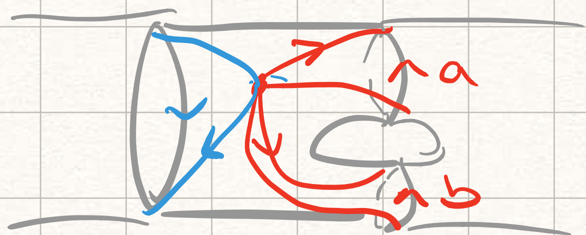
by conj. invariance.

lem 2: The Euler number $\mathcal{E}(\rho|_{\pi_1 \Sigma_g})$ is either 0 or $\pm(2g-2)$.

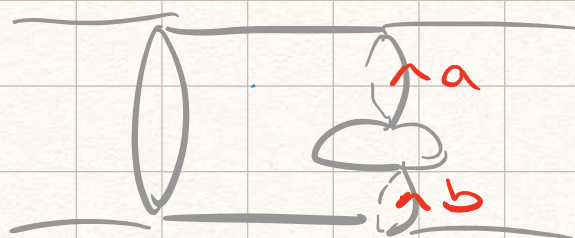
Def: Suppose have

$\rho : \pi_1 \Sigma_g \to \mathrm{Homeo}^+(S^1)$.

Take pants decomp of $\pi_1 \Sigma_g$ for pant P



$a^{-1}b^{-1}$ basically this:



then

$$\varepsilon_P(\rho) := \mathrm{rot}(\tilde{\rho}(a)) + \mathrm{rot}(\tilde{\rho}(b)) + \mathrm{rot}(\tilde{\rho}(a)^{-1}\tilde{\rho}(b)^{-1})$$

$$\varepsilon(\rho) := \sum_{P \text{ in decomp}} \varepsilon_P(\rho)$$

exc: Indep of pants decomp.

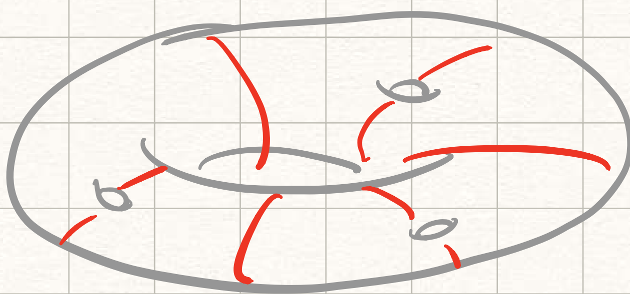
FACT: $|\mathcal{E}_P(\rho) = 1| \leq 1$ "quasi-morphism"

Pf of lem 2:

MCG $\circlearrowleft$ on P's.

P's $\to$ P's.

P



Pick these P's.

$\mathcal{E}_P(\rho) = 0 \mod \mathbb{Z}$ b/c non-sep (lem 1).

$\Rightarrow \pm 1$ or 0 by FACT.

Every pant here can be taken to another by an automorphism, so get same value on each pant $\Rightarrow$ get 0 or $\pm 1(2g-2)$ in total.

Thm: If $\mathcal{E}(\rho|_{\pi_1 \Sigma_g})$ is $\pm(2g-2)$, then $\rho|_{\pi_1\Sigma_g}$ is semi-conj to the std boundary action.

Using the fact that $\pi_1 \Sigma_g$ is normal in $\mathrm{Aut}^+ \Sigma_g$, can show ρ is itself semi-conj to the std boundary action.

(YAY!).

Remains to show that if $\varepsilon(\rho|_{\pi_1 \Sigma_g}) = 0$, then ρ is trivial.

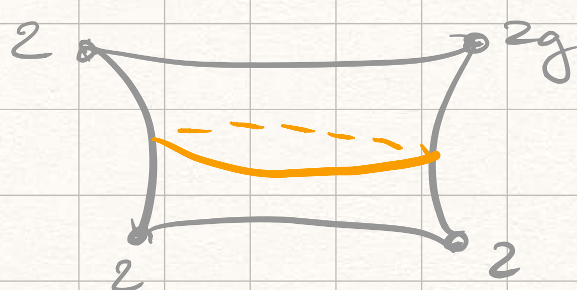
Study instead Aeelbgrp of $\mathrm{Aut}^+(\pi_1 \Sigma_g)$.

$$\Delta = \left\langle a, b, c, d \;\middle|\; \begin{array}{l} a^2 = b^2 = c^2 = d^{2g} \\ = abcd = 1 \end{array} \right\rangle$$

$$\downarrow$$

$$D_{4g}$$

w/ kernel $\pi_1 \Sigma_g$.



Euler # applies to $\rho|_\Delta$ also, and take a value here that is a multiple of $\mathcal{E}(\rho|_{\pi_1\Sigma_g})$.

It is of the form

$$\mathcal{E}(\rho|_\Delta) = \mathrm{rot}(\rho(a)) + \mathrm{rot}(\rho(b)) + \underbrace{\mathrm{rot}(\rho(c))}_{\frac{1}{2}\mathbb{Z}} + \mathrm{rot}(\rho(d)) + \underbrace{\text{integer}}_{\frac{1}{2}\mathbb{Z}}$$

$$\Rightarrow \mathrm{rot}(\rho(d)) \in \tfrac{1}{2}\mathbb{Z}$$

$$\Rightarrow (\rho(d))^2 = \mathrm{id}.$$

<u>ex</u>: d^2 normally generates $MCG(\Sigma_{g,\bullet})$ $\to$ trivial.

(exc).

$\square$