Hyperbolic Geometry

1) Riemannian Geom

X = smooth manifold

TX = tangent space

$T^2X \ni (x, u, v)$, $x \in X$

$u, v \in T_x X$

Riemannian metric

$m : T^2X \to \mathbb{R}$ C^∞

st $\forall x \in X$

$m_x : T_x X \times T_x X \to \mathbb{R}$

scalar product.

Length of path

$c : [a,b] \xrightarrow[C^1]{} X$

$$\text{length}(c) = \int_{t=a}^{t=b} \sqrt{m_{c(t)} (\dot c(t), \dot c(t))}\, dt$$

at $c(t)$ speed

distance on X

$$d(x,y) = \inf_{\substack{c : [a,b] \to X \\ c(a) = x,\ c(b) = y}} \text{length}(c) .$$

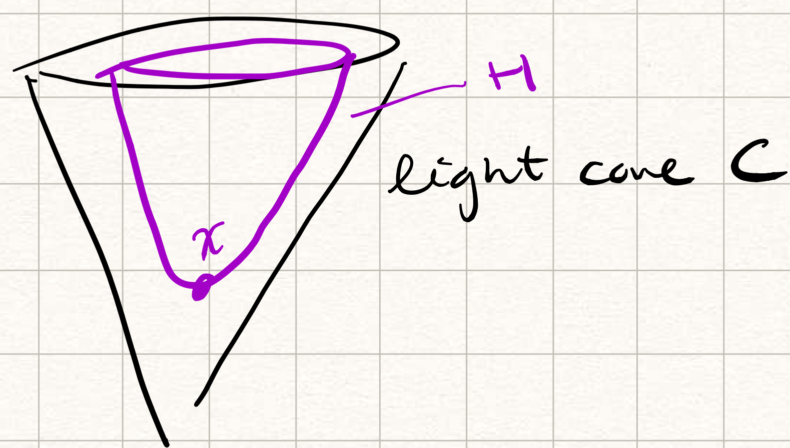
2) 3 models of hyp. spaces

1) ½ - hyperboloid

$$g(x) = -x_0^2 + x_1^2 + \dots + x_d^2$$
on $\mathbb{R}^{d+1}$

$C =$ light cone
$= \{x \in \mathbb{R}^{d+1} \mid g(x) = 0,\ x_0 > 0\}$

$H = \{x \in \mathbb{R}^{d+1} \mid g(x) = -1,\ x_0 > 0\}$



$x \in H$,
$x^\perp = T_x H$
$q|_{T_x H}$ is definite pos.

put on H the Riemannian metric
given on each tangent $T_x H$

scalar product $g|_{T_x H}$

2) Conformal Ball

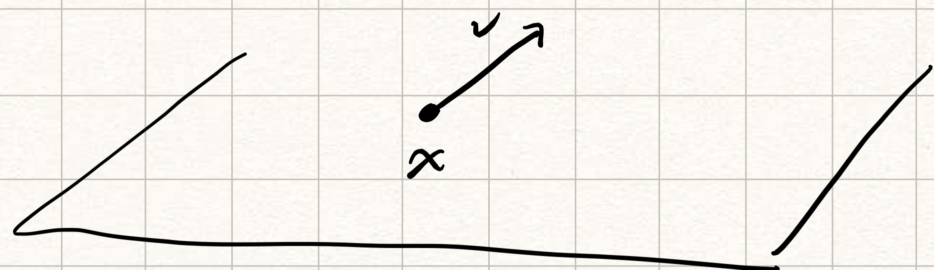
$X = B^d = \{x \in \mathbb{R}^d \mid x_1^2 + \dots + x_d^2 < 1\}$

$x \in B^d, \quad v \in \mathbb{R}^d$

$$|v|_{hyp} = \frac{2\,|v|_{euclidean}}{1 - |x|^2}$$

3) Upper half-space model

$\mathcal{U}^d = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_d > 0\}$



$x \in \mathcal{U}^d, \; v \in T_x \mathcal{U}^d = \mathbb{R}^d$

$$|v|_{hyp} = \frac{|v|_{eucl}}{x_d}$$

Prop: 1), 2), 3) are all the same. (isometric).

Exceptional isomorphism

$\mathrm{Isom}^+(\mathcal{U}^2) = PSL_2(\mathbb{R})$

Q: What do we see in each model?

- $O_{d,1}(\mathbb{R}) = O(q)$ acts by isometries,
  $= \{g \in GL_{d+1}(\mathbb{R}) \mid q \circ g = q\}$
  q quad. form

ORTHOGONAL TRANSFORM

- $\dim O_{d,1}(\mathbb{R}) = \frac{d(d+1)}{2}$
- $O_d(\mathbb{R})$ acts by isom on B^d fixing 0.
- $\mathrm{Sim}(\mathbb{R}^{d-1}) = \{$similarity of $\mathbb{R}^d$ that preserves $\mathbb{R}^{d-1} \times \{0\}\}$
  isom × scaling

  act by isometries on $\mathcal{U}^d$ fixing $\infty \in \partial \mathcal{U}^d$.

Boundary

B^d, $\partial B^d = S^{d-1}$

$\partial \mathcal{U}^d = (\mathbb{R}^{d-1} \times \{0\}) \cup \{\infty\}$

$\mathcal{U}^d \subset \underbrace{\mathbb{R}^d \cup \{\infty\}}_{\text{Stereographic proj.}}$

$\partial \mathcal{U}^d = (\mathbb{R}^{d-1} \times \{0\}) \cup \{\infty\}$

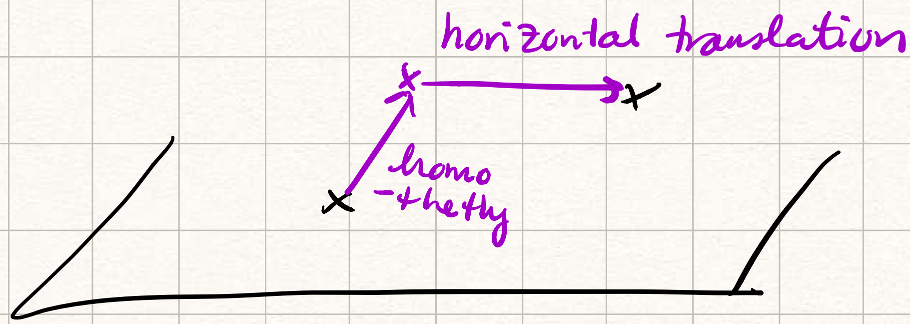
Prop: i) $\mathrm{Isom}(\mathbb{H}^d) \curvearrowright \mathbb{H}^d$ is transitive and

ii) $\mathrm{Isom}(\mathbb{H}^d) \curvearrowright \partial \mathbb{H}^d$ is 3-transitive.

Pf: i) since $\mathrm{Isom}(\mathcal{U}^d) \supset \mathrm{Sim}(\mathbb{R}^{d-1}) \underset{\text{trans}}{\curvearrowright} \mathcal{U}^d$ (A)

ii) since $\mathrm{Isom}(\mathbb{B}^d) \supset O_d(\mathbb{R}) \curvearrowright \partial \mathbb{B}^d$ is transitive.

A:



· $\mathrm{Stab}_\infty \leq \mathrm{Isom}(\mathcal{U}^d) \curvearrowright \underbrace{\partial\mathcal{U}^d \setminus \{\infty\}}_{=\mathbb{R}^{d-1}}$ is trans.

thanks to horizontal translation.

· $\underset{\text{linear sim of } \mathbb{R}^{d-1}}{\overset{\parallel}{\mathrm{Stab}_{\infty,0}}} \leq \mathrm{Isom}(\mathcal{U}^d)$

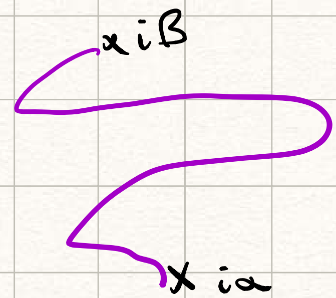
$\curvearrowright \underbrace{\partial\mathcal{U}^d \setminus \{0,\infty\}}_{=\mathbb{R}^{d-1}\setminus\{0\}}$

this act. is indeed transitive.

3) The Geodesics

a) Particular case

<u>Prop</u>: In $\mathcal{U}^2$, there is one and <u>only</u> one geodesic b/w $i\alpha \in \mathbb{C}$ & $i\beta \in \mathbb{C}$. w/ $0 < \alpha < \beta$ which is the euclidean segment.



$$\int_a^b \frac{|\dot{c}(t)|}{c(t)}\, dt$$

$$= \int_a^b \frac{\sqrt{\dot{x}^2(t) + \dot{y}^2(t)}}{y(t)}\, dt$$

$$\geqslant \int \frac{|\dot{y}(t)|}{y(t)}\, dt \qquad \Bigg| \begin{array}{l} \text{w/ equality} \\ \Leftrightarrow \quad \dot{x} = 0 \\ \Leftrightarrow \quad x = ct. \end{array}$$

$$\geqslant \int_a^b \frac{\dot{y}(t)}{y(t)} \qquad \Bigg| \begin{array}{l} \text{w/ equality} \\ \Leftrightarrow \ \dot{y} \geqslant 0 \\ \Leftrightarrow \ y \text{ incr.} \end{array}$$

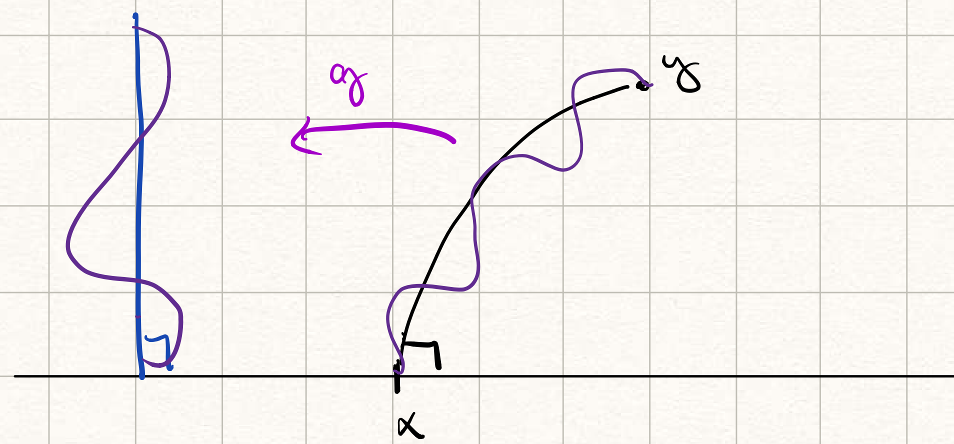
$$= \ln(\beta/\alpha)$$

<u>ie</u> the support of the geod is the Euclidean segment.

b) General case

Prop: There is one and only one geod b/w $x, y \in \mathcal{U}^2$,
which is the circ-line $\mathcal{C}$ orth. to $\partial\mathcal{U}^2$ w/ $x, y \in \mathcal{C}$.

Pf:



$g(x) = 0, \quad g \in PSL_2(\mathbb{R})$

then use a).

Prop: In $\mathcal{U}^d, \mathbb{B}^d$ geods will be circ-line boundary,
in hyperbaloid model,
geodesics are intersection 2-plane Π w/ H.

4) Hyperbolic Space is Gromov Hyperbolic.

Gromov hyp = ∂-hyp.