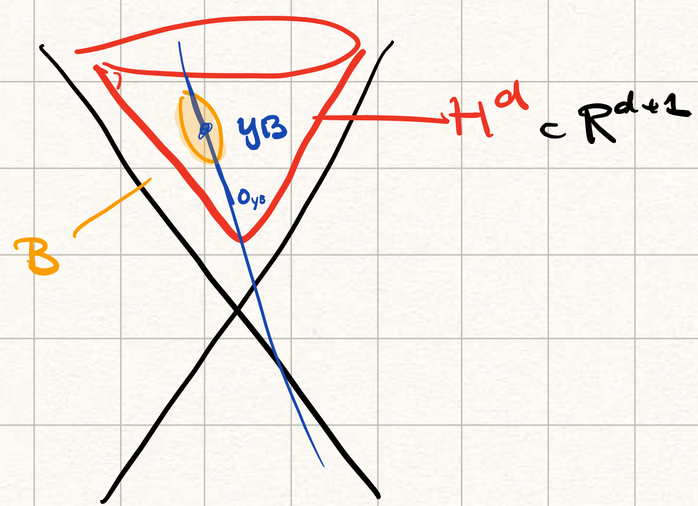
Classifying isoms

Lemma 1: (existence of center)
Let B be a bounded convex subset of $\mathbb{H}^d$, then $\exists x_B \in B$ $\forall g \in \mathrm{Isom}(\mathbb{H}^d)$,
$gB = B$,
$g x_B = x_B$.

↑ center of B

Pf 1) CAT(0) geom
↳ circumcenter of B

Pf 2)



C = convex hull $(B) \subset \mathbb{R}^{d+1}$
y_B = barycenter of C
$x_B := (0\, y_B) \cap \mathbb{H}^d$

$\mathrm{Isom}(\mathbb{H}^d) = O(d,1) \leq GL_{d+1}(\mathbb{R})$
preserves barycenter
(center of mass).

Lemma 2 (existence of fixed pts)

$g \in \mathrm{Isom}(\mathbb{H}^d)$, then $\mathrm{Fix}(g) \neq \emptyset$

$\cap$

$\mathbb{H}^d \cup \partial \mathbb{H}^d$

Pf 1 : Brouwer Thm.

Pf 2 : Soit $x \in \mathbb{H}^d$

$(x_n) = (g^n x)_{n \in \mathbb{Z}}$ is bounded

convex hull of x_n

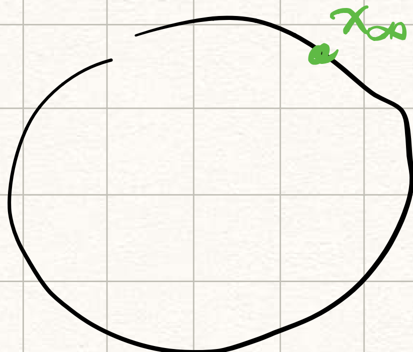
→ center → fix pt for g

↪ $\mathbb{H}^d$ case.

else: ⇒ hull not bonded.

↘ go to $\partial\mathbb{H}^d$ case.

$\Rightarrow \exists (n_i)$ st $g^{n_i} x \xrightarrow[i \to \infty]{} x_\infty \in \partial \mathbb{H}^d$



x_∞

$g(x_\infty) = x_\infty$

$y_{n_i} = g^{n_i}(gx)$

$d(x_{n_i}, y_{n_i}) = d(g^{n_i} x, g^{n_i} g x)$

$= d(x, gx)$

$\leq \mathbb{R}$

$\Rightarrow$ fellow-travel. lemma

$$g^{n_i}(gx) \longrightarrow x_\infty$$
$$\|$$
$$g(\underbrace{g^{n_i}x}_{\to\, x_\infty})$$
$$g(x_\infty) = x_\infty$$

Pf of classification

Soit $g \in \mathrm{Isom}(\mathbb{H}^d)$

$\rightsquigarrow$ g has <u>bounded orbit</u>

$\rightsquigarrow$ g elliptic

$\tau(g) = 0 \Rightarrow$ inf is min

$\mathrm{Fix}(g) \cap \mathbb{H}^d \neq \emptyset$

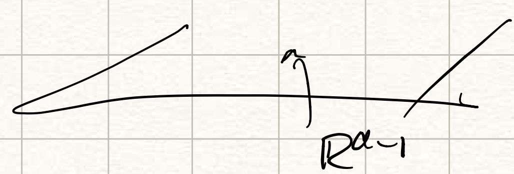
($\rightsquigarrow$ <u>all</u> orb are bounded)

g <u>has no bounded orbit</u>

$\rightsquigarrow \exists\, p \in \partial\mathbb{H}^d$ st $g(p) = p$

$\mathbb{H}^d = \mathcal{U}^d$, $p = \infty$

$\partial\mathcal{U}^d \setminus \{\infty\}$

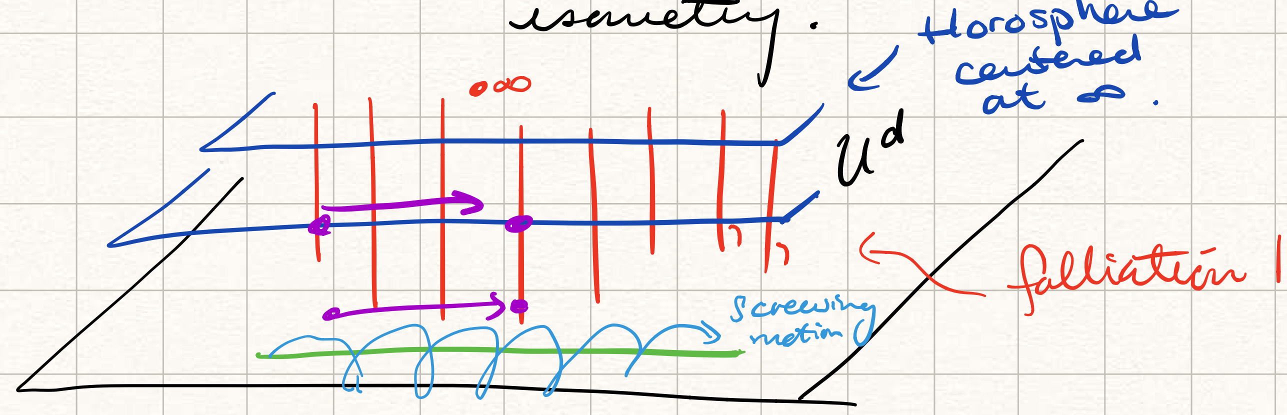


$\mathrm{Stab}_\infty = \mathrm{Sim}(\mathbb{R}^{d-1})$

$\in$

g

case 1 : $g \curvearrowright \partial \mathcal{U}^d \setminus \{\infty\}$ is an isometry.



- g permute the geodesics containing ∞.
- g preserves each horizontal hyperplanes $\mathbb{R}^{d-1}$

$\Rightarrow g \curvearrowright \partial \mathcal{U}^d \setminus \{\infty\}$ has no fixed pt

Recall : g is a Euclidean isom on $\mathbb{R}^{d-1}$

$\hookrightarrow \exists!\ (t, A)$ where

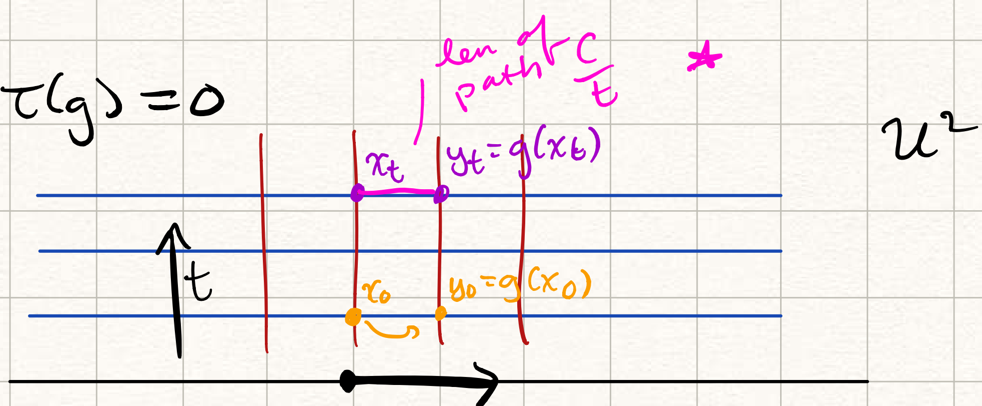
t is a translation,

A is an isom w/ a fixed pt

and $g = t \circ A = A \circ t$.

PT : parabolic element = translation + bounded transfo (A).

Claim : $\tau(g) = 0$



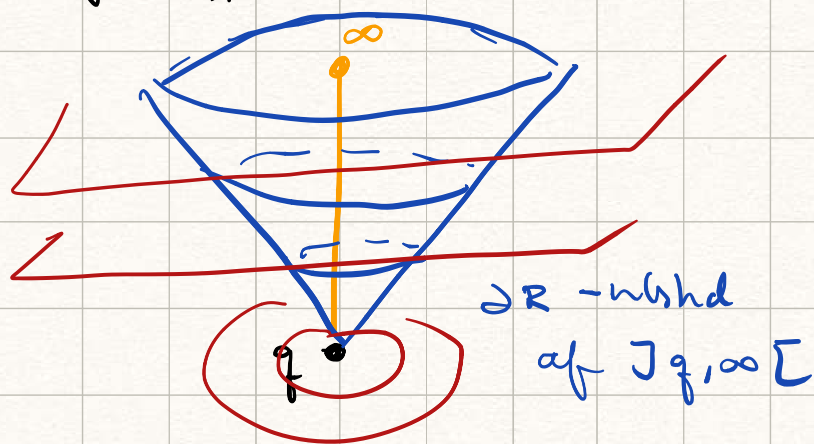
$\ldots \text{leng}(c) = \int_a \frac{\ldots}{t}\, dx.$

$$d(x_t, y_t) \xrightarrow[t \to \infty]{} 0$$
$$\leq \frac{c}{t}$$

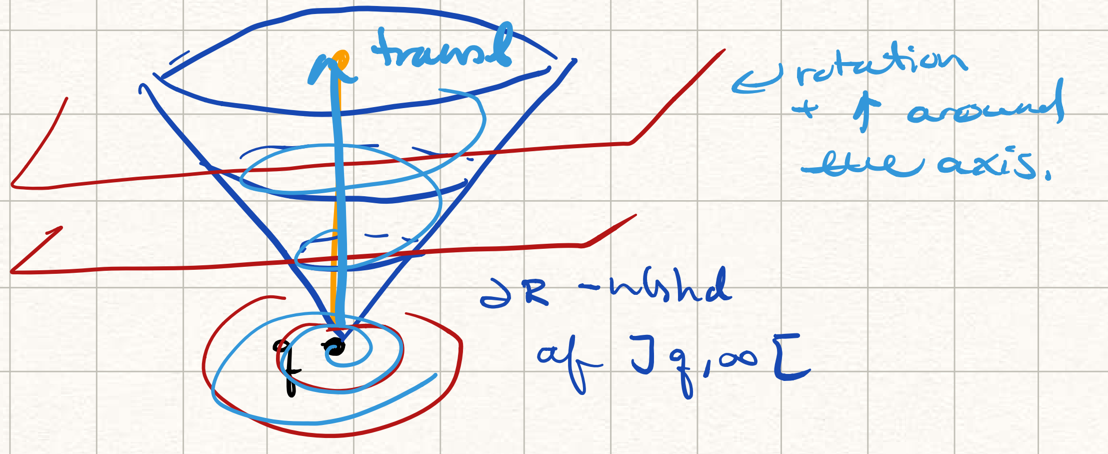
<u>case 2</u>: $g \circlearrowright \partial \mathcal{U}^d \setminus \{\infty\}$ is a similarity of ratio $\lambda \neq 1$, $\lambda > 0$

$\to g \circlearrowright \mathbb{R}^{d-1}$ has a unique fixed pt q.

$$q = \lim_n \overset{\text{sim}}{f^{-n}}(x) \quad \text{of any} \; (\lambda > 1)$$



- g preserves the foliation of $\mathbb{H}^d$ by the horospheres centered at ∞.
- g preserves each R-nbhd of $]q, \infty[$.



g is a similarity of $\mathbb{R}^{d-1}$ of ratio $\lambda \neq 1$

$\exists!\ (h_\lambda, A)$ st

$$g = h_\lambda \circ A = A \circ h_\lambda$$

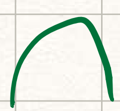
h_λ = homothety ratio λ

A = isom that fixes q.

$$\tau(g) = d(x, gx)$$

$\Leftrightarrow \quad x \in \,]\infty, q[$

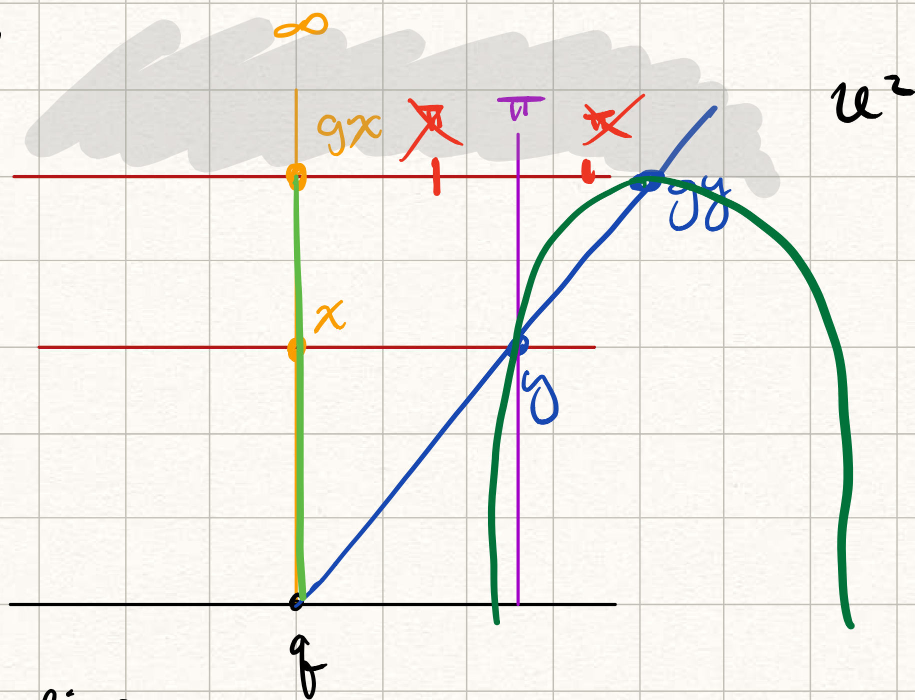
trying to show that shorter than



$\Rightarrow$ project y on convex,

$\rightarrow$ it's unique so must be on line.

o/w non-unique.



$$d(x, gx) = d(y, \pi) < d(y, gy)$$

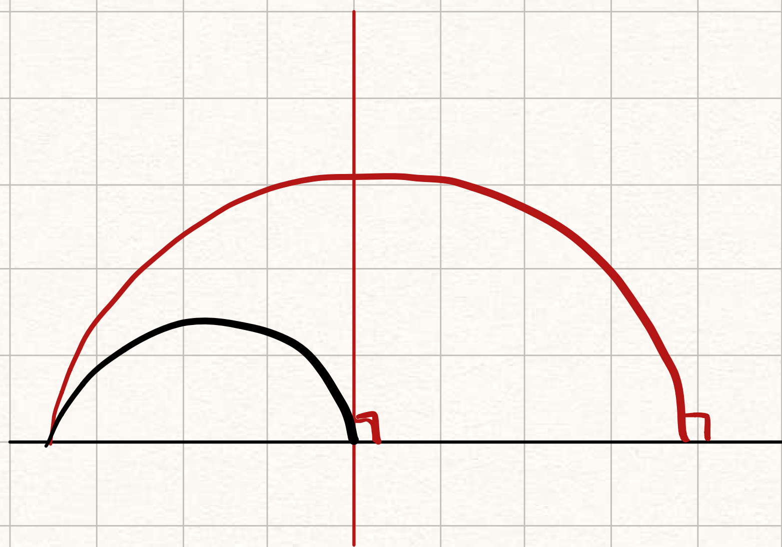
exc : Draw pict in other model.

5^{th} Euclidean Postulate

Unique parallel.

- S^2 violates b/c no para.

$\mathbb{H}^2$



so many k?. Good.