Thm: $\mathbb{H}^d$ is Gromov hyp.

Fact: 3pts are on a hyp. plane
(take 1 pt, expand $\mathbb{B}^d$ to $\cap$ w/ other two).

Assume $d=2$, & $\mathbb{H}^d$ NOT Gromov hyp.

$\Rightarrow \exists (x_n), (y_n), (z_n) \in \mathbb{H}^2$

$\exists (u_n)$ st $u_n \in [x_n, y_n]$

$$d(u_n, [y_n, z_n] \cup [z_n, x_n]) \to \infty$$

Since $\mathbb{H}^2$ is homogeneous, we may assume that $u_n = ct = u$

$\exists g_n$ isom $\Rightarrow$ $g_n u_n = u$ then apply to x_n, y_n, z_n.

$\mathbb{H}^2 = B^2$, $\overline{\mathbb{H}}^2 = B^2 \cup \partial B^2$

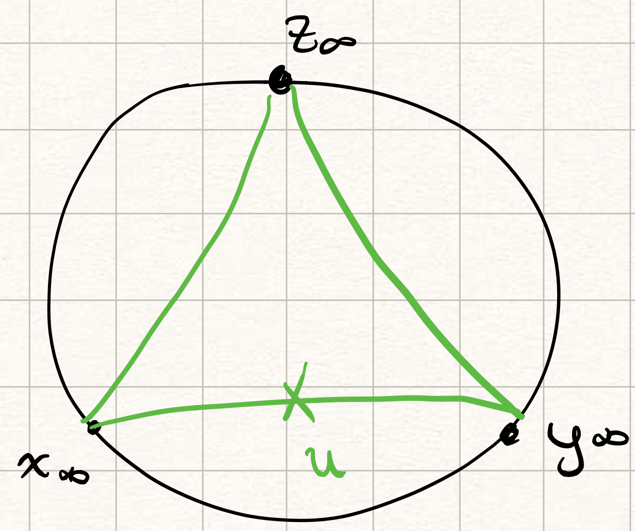
We may assume $(x_n), (y_n), (z_n)$ CV

$d(u, x_n) \to \infty$

$d(u, z_n) \to \infty$

$d(u, y_n) \to \infty$

Hence: $x_\infty, y_\infty, z_\infty \in \partial \mathbb{H}^2$



But $d(u, \,]x_\infty, z_\infty[\,\cup\,]z_\infty, y_\infty[\,) < \infty$.

contra

$\square$

The sphere of $\mathbb{H}^d$

Thm: Every metric sphere of $\mathbb{H}^d = \mathcal{U}^d$ or B^d are usual sphere.

upper half-space model

ball

Pf: $\mathbb{H}^d$ is homogenous

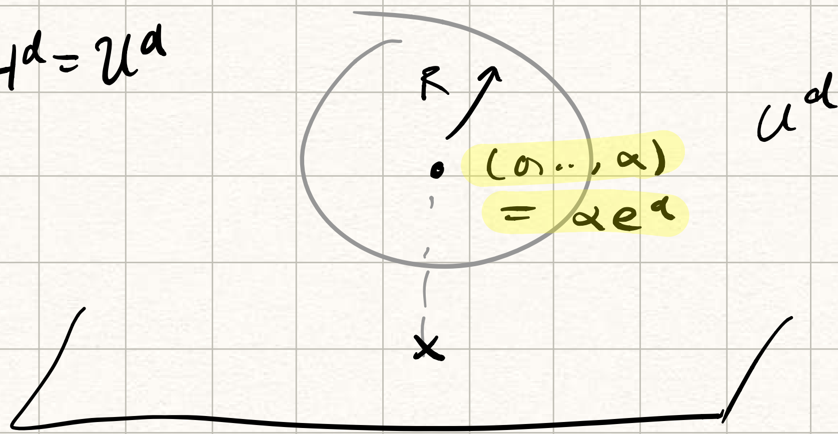
$\Rightarrow$ $\mathrm{Isom}(\mathbb{H}^d)$ action on all the ball of radius a fixed R is transitive.

WTS: In B^d, $S_{\mathbb{H}}(\text{origin}, R) = S_{\mathbb{E}}(\text{origin}, R')$

sphere in hyp. metric

exc: If you look at $\mathrm{Isom}(\mathbb{B}^d)$ or $\mathrm{Isom}(\mathcal{U}^d)$ acts on $\mathbb{R}^d \cup \{\infty\}$ preserves the set of hyperplane – sphere. $\square$

Lemma: $\mathbb{H}^d = \mathcal{U}^d$

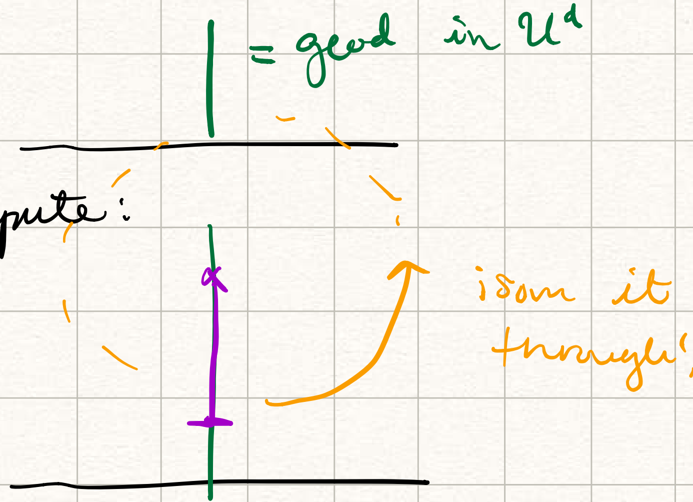


we have:

$$S_{\mathbb{H}}(\alpha e_d, R) = S_{\mathbb{E}}(\alpha \cosh(R)\, e^d, \alpha \sinh(R))$$

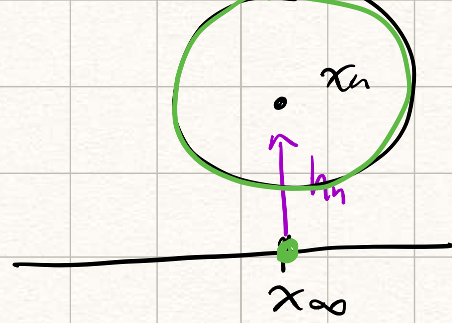
hint: Use that

compute:



Rk: $x_n \in \mathcal{U}^2$, $\to x_\infty \in \partial \mathcal{U}^2$

$x_\infty \neq \infty$



PT the ball shrinks to a pt in boundary.

Fix $R>0$, $h_n \sinh(R) \xrightarrow[n\to\infty]{} 0$

Lemma (Fellow-traveler)

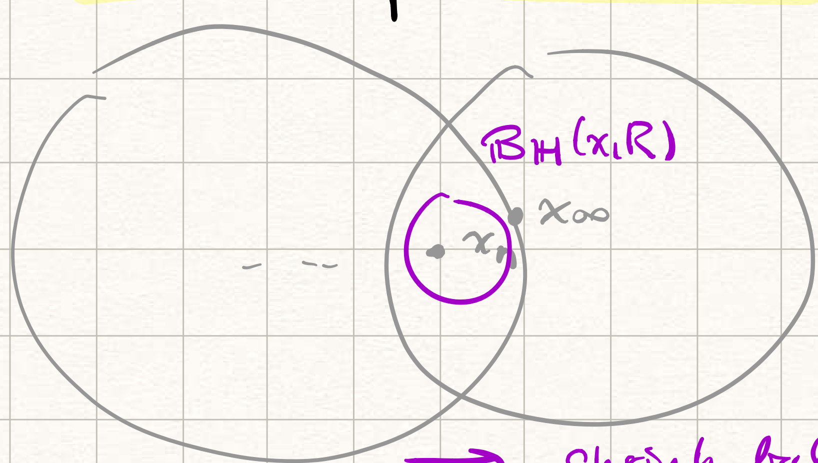
Let $(x_n), (y_n) \in \mathbb{H}^d$ st

- $x_n \longrightarrow x_\infty \in \partial H^d$,
- $\exists R>0$ st $d(x_n, y_n) < R$ $\forall n$.

Then: $y_n \xrightarrow[n\to\infty]{} x_\infty$

PT: two path at ct distance converge to same pt on $\partial \mathbb{H}^2$

Pf:



$\longrightarrow$ shrink ball.

3. Classification of isometry

Def: $g \in \mathrm{Isom}(\mathbb{H}^d)$,
let $\mathrm{Fix}(g) = \{x \in \mathbb{H}^d \mid g(x) = x\}$
let $\tau(g) = \inf_{x \in \mathbb{H}^d} d(x, gx)$

= translation length.

Thm: Let $g \in \mathrm{Isom}(\mathbb{H}^d)$ we have 3 cases:

1. $\mathrm{Fix}(g) \cap \mathbb{H}^d \neq \emptyset$ and inf is achieved $\to$ g is elliptic.
2. $\mathrm{Fix}(g) \cap \mathbb{H}^d = \emptyset$ and $\mathrm{Fix}(g) \cap \partial \mathbb{H}^2 = 1$, g is parabolic.
3. $\mathrm{Fix}(g) \cap \mathbb{H}^d = \emptyset$ and $\mathrm{Fix}(g) \cap \partial \mathbb{H}^2$ $\to$ g is hyperbolic.

More precisely,

1. If g is elliptic,
   - then $\tau(g) = 0$, inf is a min,
   - every orbit of $g \curvearrowright \mathbb{H}^2$ is bounded

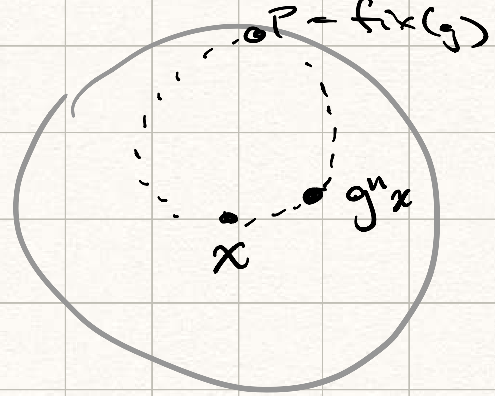
2. If g is parabolic, then

- $\tau(g) = 0$, inf is NOT achieved,
- no orbit of $g \curvearrowright \mathbb{H}^d$ is bounded.

all orbits + or − are converging to p,

$g^n x \longrightarrow p$

$\forall x \in \mathbb{H}^d$, $n \to \pm\infty$



3. If g is hyperbolic,

$\tau(g) > 0$, inf is a min

min set $(g) = \{x \in \mathbb{H}^d \mid d(x, gx) = \tau(g)\}$

is the geodesic joining the two fixed pts, p^+, p^-.

→ no orbit of $g \curvearrowright \mathbb{H}^d$ is bounded

→ $\forall x \in \mathbb{H}^d$

$g^n x \longrightarrow p^+$ attractive fix pt

$g^{-n} x \longrightarrow p^-$ repulsive fix pt