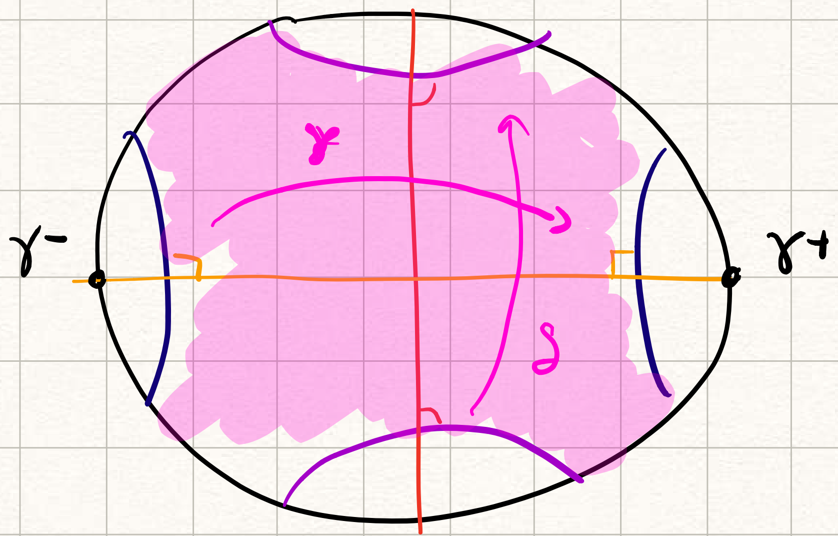
Schottky - Borel
Coxeter - Poincaré

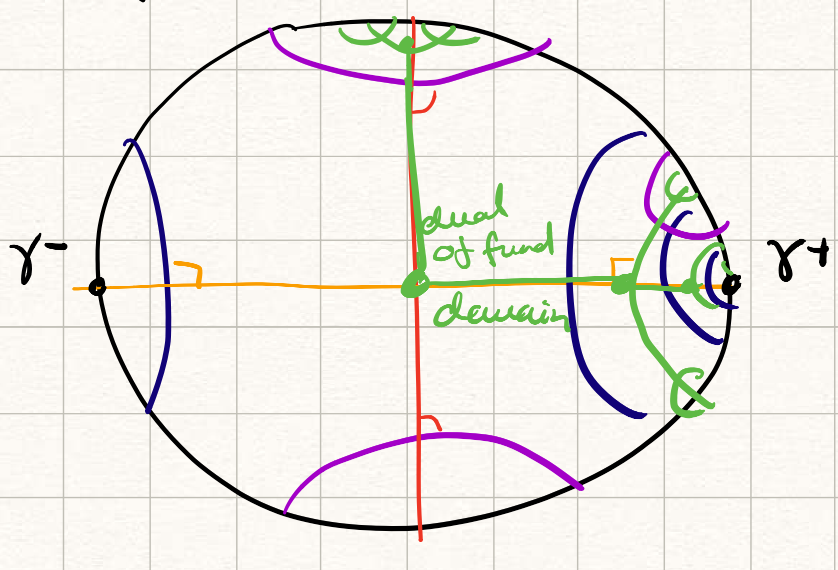
I - Schottky



$\exists!\ \delta$ sending) $\to$ (

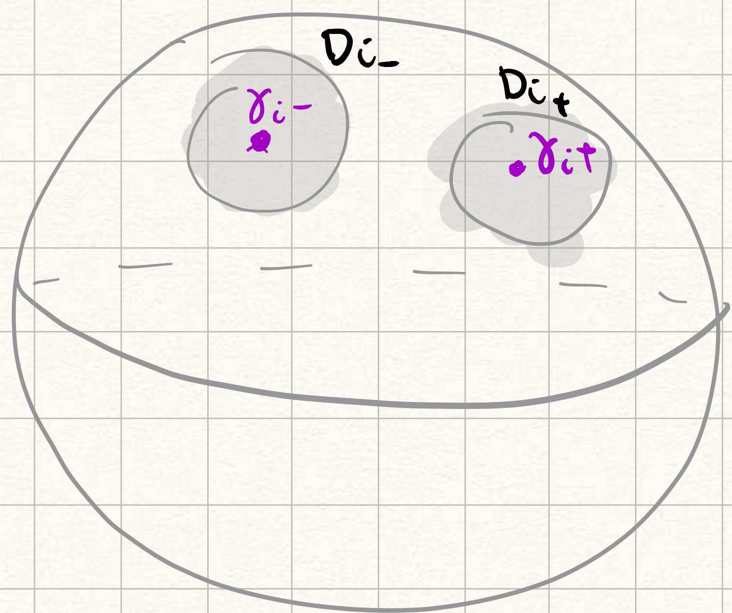
$!\ \gamma$

Thm. $\langle \gamma, \delta \rangle \leq \mathrm{Isom}(\mathbb{H}^2)$ is free, discrete and P fundamental domain.



Thm: Let D_i^+, D_i^- be an open ball in $\partial \mathbb{H}^d$ st $\overline{D_i^\pm} \cap \overline{D_j^\pm} = \emptyset$.

Take $\gamma_i - \mathrm{Isom}^+_{\overline{D_i}}(\mathbb{H}^d)$



γ_i^-, γ_i^+ attractors of hyp. elmt.

Map outside of repulsive disk to inside of attractive.

$\gamma_i(\partial \mathbb{H}^d \setminus D_i^-) \subset D_i^+$

$\Gamma = \langle \gamma_2, \ldots, \gamma_n \rangle$

Thm: Γ is free, discrete and (exc: describe fund. domain).

Cor: Let $\Gamma \leq \mathrm{Isom}(\mathbb{H}^d)$ discrete & non-elmt,
(ie, every orbit of $\Gamma \curvearrowright \mathbb{H}^d$ is
infinite) then contain
a free grp.

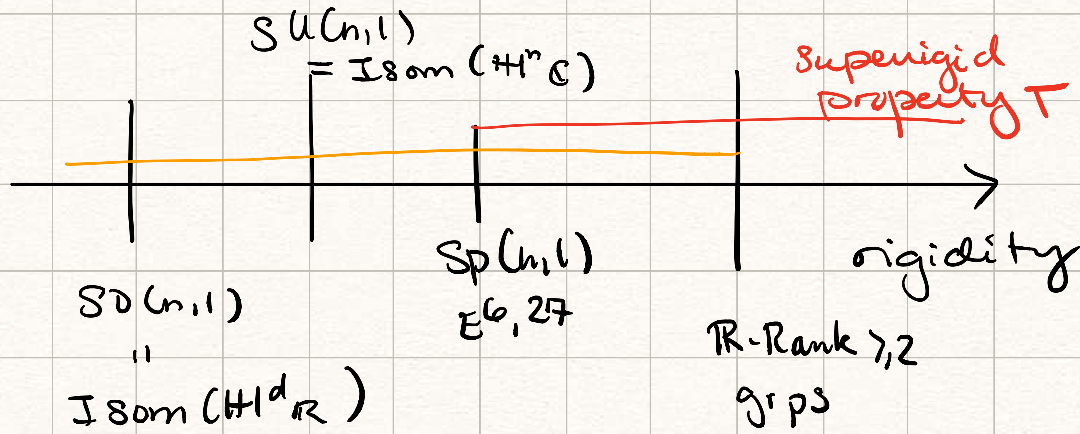
Rk: elementary discrete subgrp of
$\mathrm{Isom}(\mathbb{H}^d)$ are virt. ab.
and easy to describe.

Thm: Let γ, δ be two hyperbolic elmt
of $\mathrm{Isom}(\mathbb{H}^d)$ st
$$\# \{\gamma^+, \gamma^-, \delta^-, \delta^+\} = 4$$
\# of attract, repuls.

then $\exists N \geq 0$
$\langle \gamma^N, \delta^N \rangle$ is a free grp.

2) Arithmetic grp

Thm: (Borel - Gromov - Shapiro)
For every $d \geq 2$, there exists
infinitely many arithmetic,
(resp non-arith) which are
co-cmpt (resp non-cmpt)
in $\mathrm{Isom}(\mathbb{H}^d) = SO_{d,1}(\mathbb{R})$
(all 4 possibility $\rightsquigarrow$ can find ∞).



Rk: In $SU_{1,2}$, there is only finitely many non-arithm lattices known:

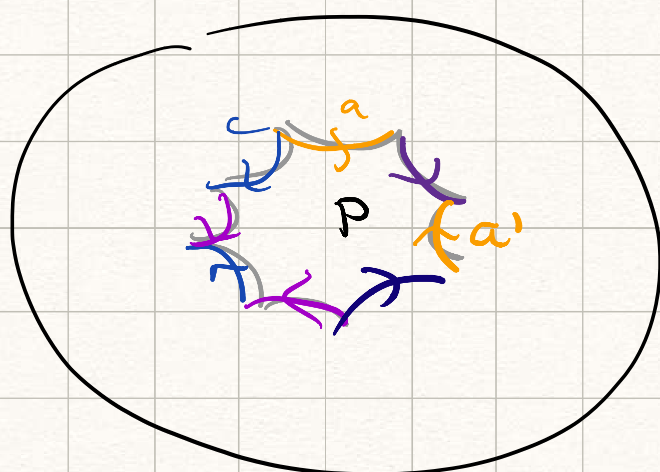
SU(3,1)

SU(4,1)

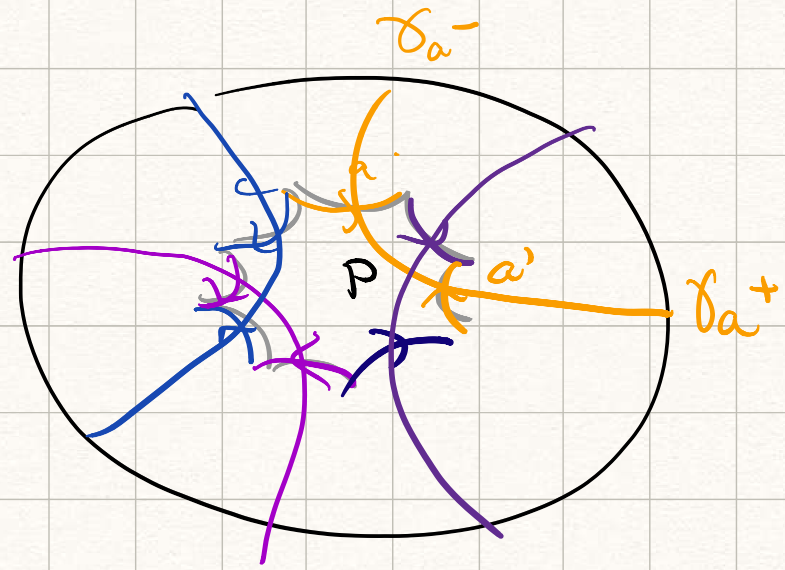
3) Smooth Poincaré

Let $g \geq 2$, let P be the reg. $4g$-gon. st

$\sum \alpha_i = 2\pi$



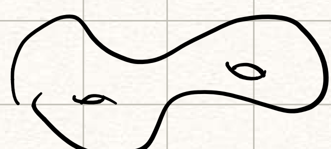
Take hyp. elmt.



Let $\gamma_a, \gamma_b, \gamma_c, \gamma_d$ be 4 hyp. isometries
send $\gamma_i(i) = i'$

$\Gamma = \langle \gamma_a, \gamma_b, \gamma_c, \gamma_d \rangle$

<u>then</u>: you get a tiling:

- $\forall \gamma \in \Gamma$, $\mathrm{int}(\gamma(P)) \cap \mathrm{int}(P) \neq \emptyset$
  $\Leftrightarrow \gamma = 1$
- $\bigcup_{\gamma \in \Gamma} \gamma(P) = \mathbb{H}^2$
- $\mathbb{H}^2 / \Gamma \underset{\text{homeo}}{\simeq}$ 

<u>Cor</u>: $\mathrm{Hyp}(\Sigma_g) \neq 0$ if $g \geqslant 2$

Rk: $\mathrm{Hyp}(\Sigma_g) \simeq \mathbb{R}^{6(g-1)}$

Thm (Mostow)

If $M = \mathbb{H}^d/\Gamma$, $M' = \mathbb{H}^d/\Gamma'$ two hyperbolic manifolds (Γ is discrete & torsion-free) of finite volume and $d \geq 3$.

If M and M' are homeomorphic, then they (Γ, Γ') are isomorphic.

If Γ, Γ' are isomorphic, then they are conjugated in $\mathrm{Isom}(\mathbb{H}^d)$.

$\rightarrow \mathrm{Hyp}(M) = \{\cdot\}$ ✗ what???

4) Polytope in Coxeter's Style

A polytope of $\mathbb{H}^d$ is a finite intersection of closed half-sp. of $\mathbb{H}^d$.

A polytope P is Coxeter if $\forall\, s, t$ facets of P,

$\dim(s \cap t) = d-2 \Rightarrow \theta(s,t) = \frac{\pi}{m}$

dihedral angle $m = \{2,3,\dots\} \cup \{\infty\}$.

Thm (Poincaré)

Let P be a cox. polytope of $\mathbb{H}^d$,

$\Gamma = \langle \sigma_s \mid s \in S \rangle$

S = set of facets of P,

σ_s = reflection across s

then: Γ is discrete, P is a fundamental domain for $\Gamma \curvearrowright \mathbb{H}^d$.

In particular, if P is compact, then Γ is a uniform lattice,

if P is of finite volume, then Γ is a lattice.