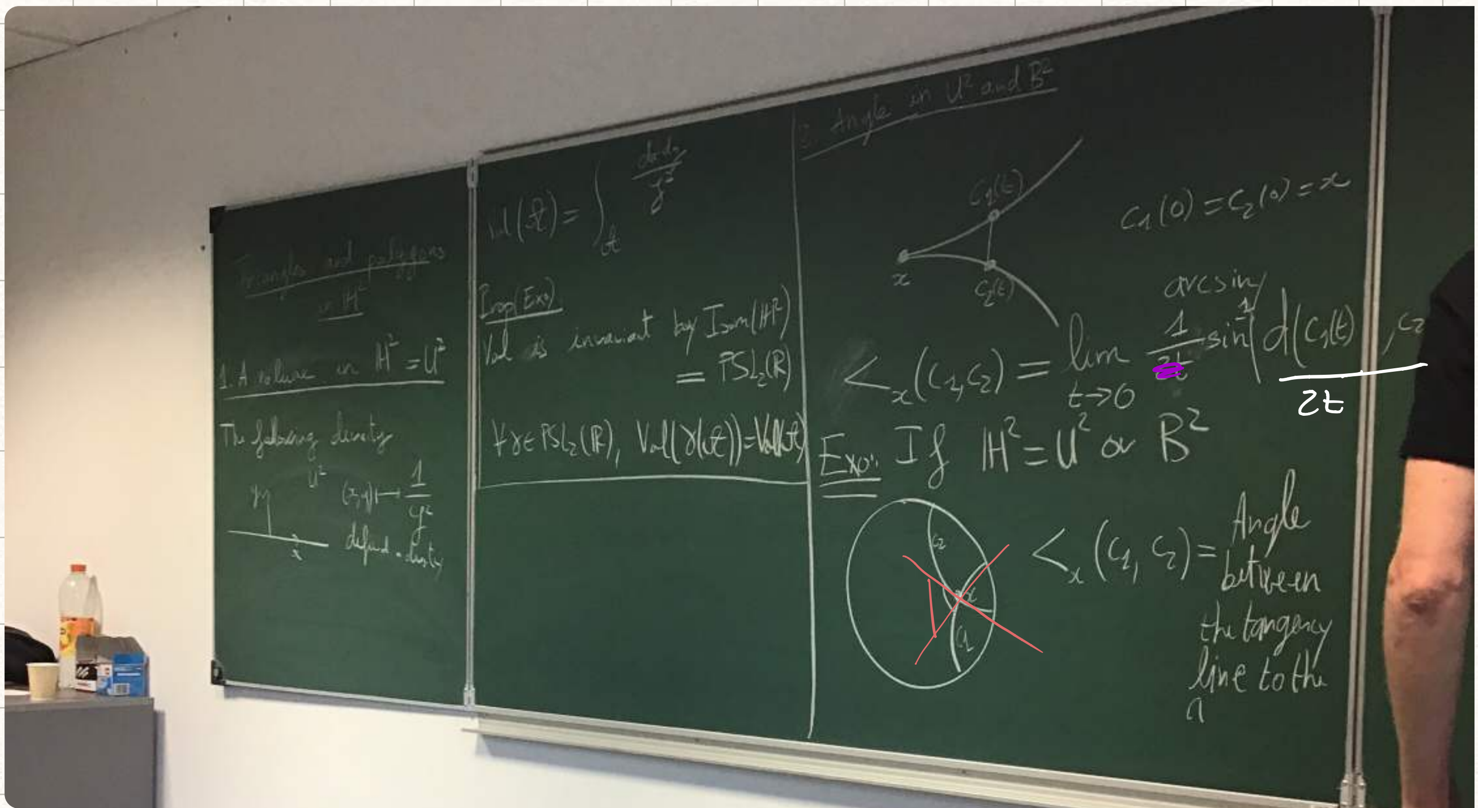
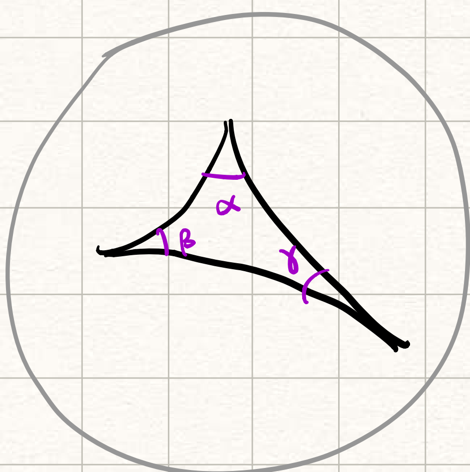



**Thm (Gauss-Bonnet)**

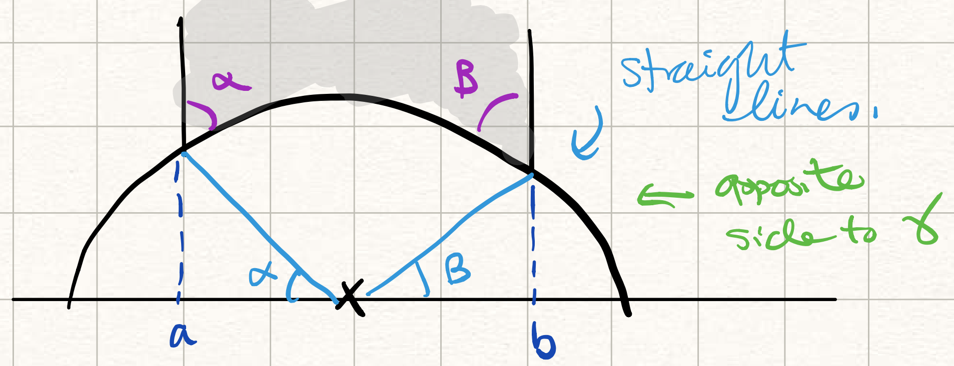
let $\Delta$ be in $\mathbb{H}^2$ (vert allow'd in $\partial(\mathbb{H}^2)$).




$$\text{area}(\Delta) = \pi - (\alpha + \beta + \gamma)$$

Cor : $\alpha + \beta + \gamma < \pi$.

Pf: $\gamma = 0 \Rightarrow$ pt on $\partial \mathcal{U}^2$ (use $X = \mathcal{U}^2$)




$$\mathrm{Vol}(\Delta) = \int_a^b \left( \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy \right)$$

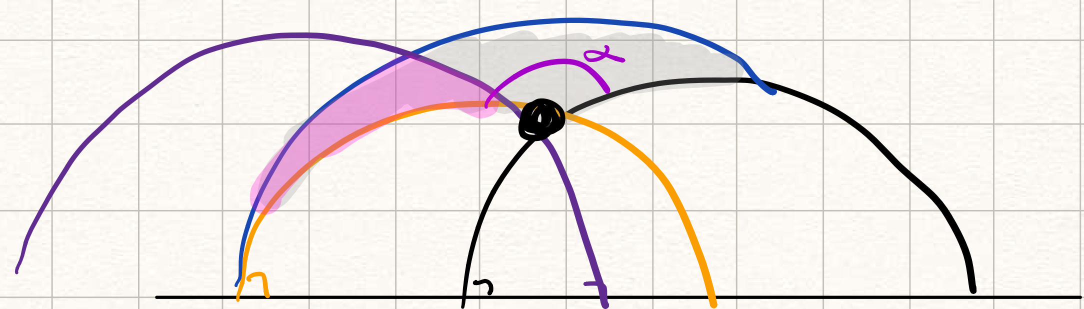
$$= \int_a^b \left[ \frac{1}{y^2} \right]_{\sqrt{1-x^2}}^{+\infty} dy$$

$$= \int_a^b \frac{1}{\sqrt{1-x^2}}\, dx$$

$\rightarrow x = \cos\theta$

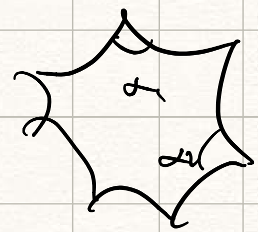
$$= \pi - (\alpha + \beta)$$

Case 2 :




$\mathrm{Vol}(\Delta) = \mathrm{Vol} + \mathrm{Vol}$

**Cor:**

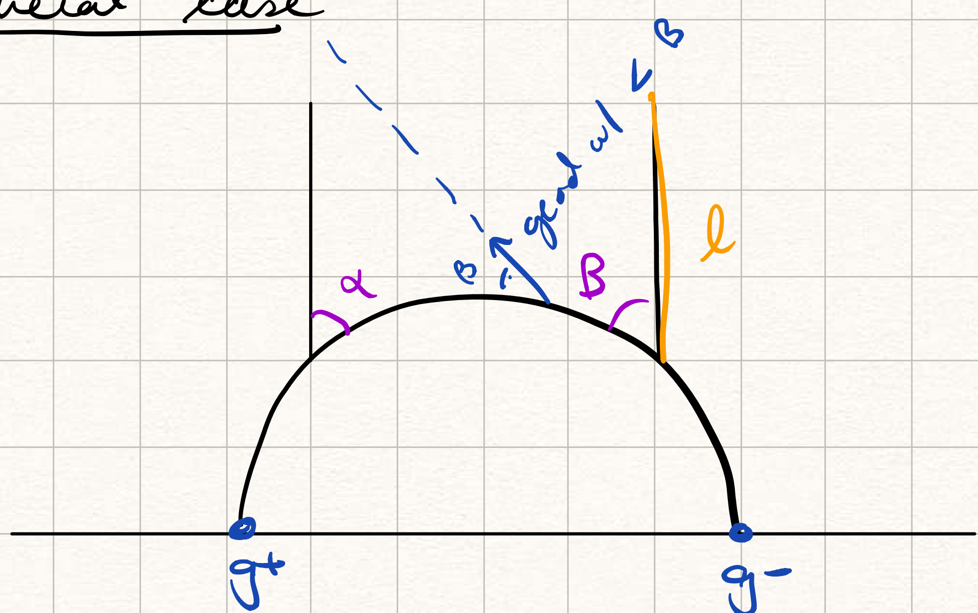
Vol(  ) $= (n-2)\pi - \sum \alpha_i$

## 3. Reciprocal

**Thm:** $\forall \alpha, \beta, \gamma \geq 0$, st $\alpha + \beta + \gamma < \pi$. There is a unique triangle in $\mathbb{H}^2$ w/ angles $\alpha, \beta, \gamma$, up to isometry.

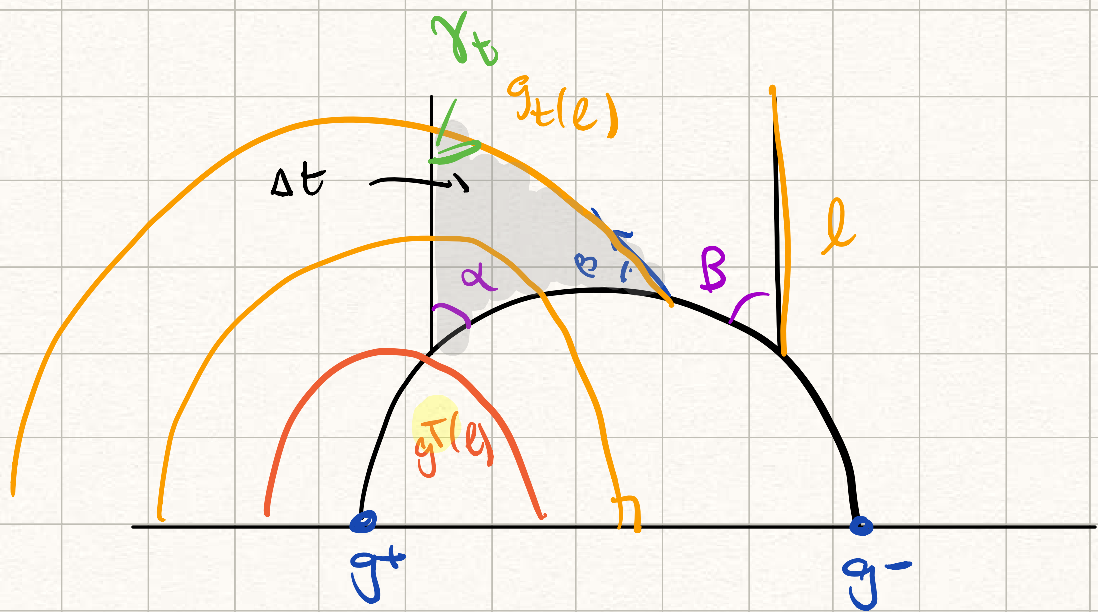
**Pf:** Look at pic of Gauss Bonnet.

General case



$\exists\ g$ hyperbolic w/ element

$$g^+ = (-1,0), \quad g^- = (1,0) \quad \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ in } PSL(2,\mathbb{R})$$




$$\begin{aligned} [0,T] &\longrightarrow \mathbb{R} \\ t &\longmapsto \gamma_t \\ 0 &\longmapsto 0 \end{aligned} \qquad \text{incr.}$$

$$\mathrm{Vol}(\Delta_t) = \pi - (\alpha + \beta + \gamma_t)$$

$$\Big\downarrow t \to T$$

$$0$$

$$\gamma_T = \pi - (\alpha + \beta)$$

so $\gamma_T > \gamma$ since $\alpha + \beta + \gamma < \pi$

by continuity, $\exists t$ st $\gamma_t = \gamma$.

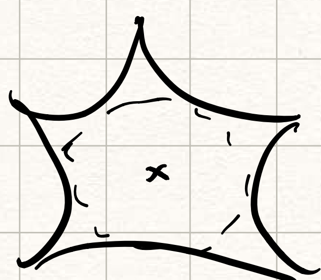
Intermediate value thm.

**Thm**: For every n-tuple, $(\alpha_1, \dots, \alpha_n)$ st $\sum \alpha_i < (n-2)\pi$.

There exists n-gon w/ angle $(\alpha_1, \dots, \alpha_n)$.

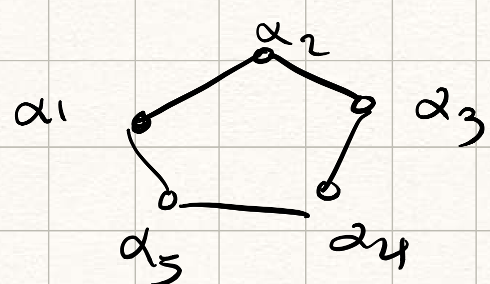
**Rk**: Unicity fails (Beardon)

**Rk**: There is a unique one w/ an inscribed circle.



$\mathcal{P}$ = the cyclic graph + label $\alpha_i$ on the vertex $i$.

$\mathcal{P} =$



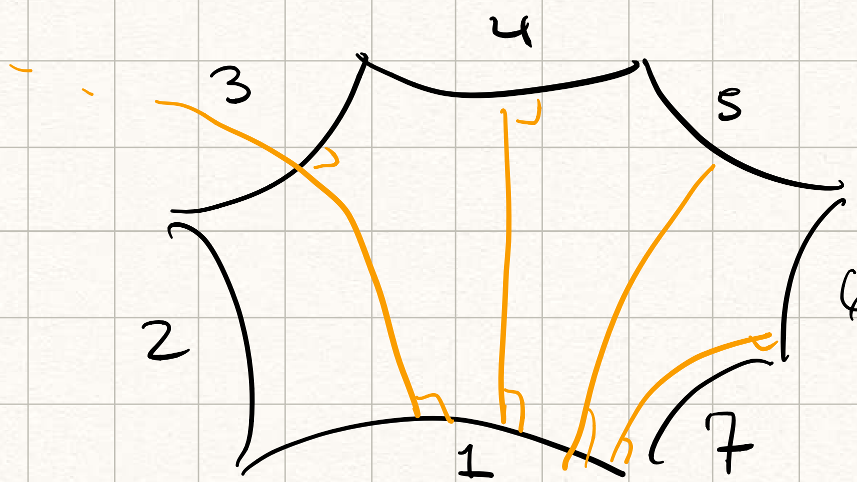

$\underbrace{H^2 \times \dots \times H^2}_{n \text{ times}}$

$$\mathrm{Hyp}(\mathcal{P}) = \left\{ \begin{array}{c} P \text{ n-gon of } \mathbb{H}^2 \\ \text{that realizes } \mathcal{P} \end{array} \right\} \Big/ \text{up to isom.}$$

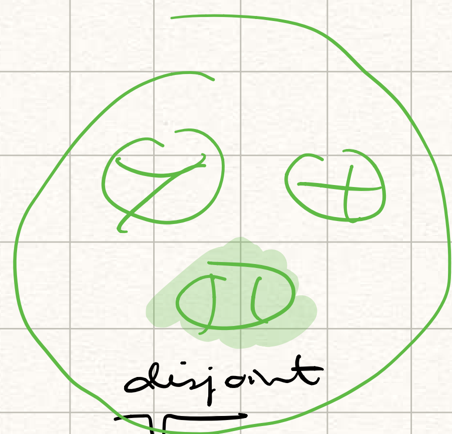
$\rightsquigarrow$ topological space.

**Thm:** $\mathrm{Hyp}(\mathcal{P}) \underset{\text{homeo}}{\cong} \mathbb{R}^{n-3}$.

**hint:**




The parameters are the length of the orange geods.




**Lemma:** If $l, l'$ are two geodesics in $\mathbb{H}^2$, there is a unique perpendicular to both.

# Polygon $\rightsquigarrow$ surfaces

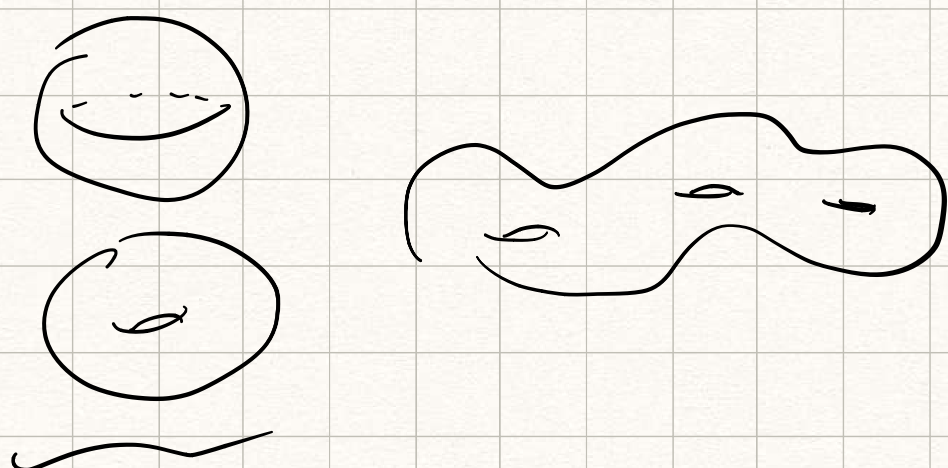
$\Sigma$ surfaces compact, orientable.

$$\mathrm{Hyp}(\Sigma) = \left\{ \begin{array}{l} \text{hyperbolic} \\ \text{structure on } \Sigma \end{array} \right\}$$

$$\Gamma = \pi_1(\Sigma)$$
$$= \{ \rho : \Gamma \longrightarrow \mathrm{Isom}(\mathbb{H}^2) \mid \mathbb{H}^2/\Gamma \simeq \Sigma \} \,/\text{up to conjugation}$$

<u>Thm</u>: (Fricke-Klein, Teichmüller)
$\mathrm{Hyp}(\Sigma) \neq \emptyset \iff \text{genus } \Sigma \geq 2$.

$\Sigma$ :



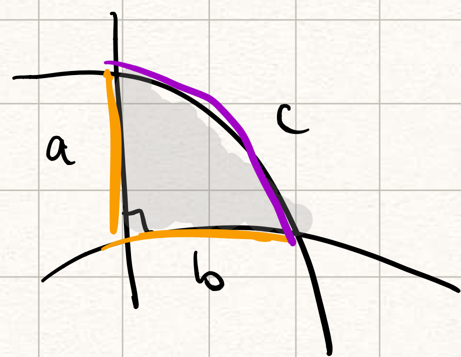
$g < 2$

$\Sigma$ of genus $\geq 2$,
$$\mathrm{Hyp}(\Sigma) \cong \mathbb{R}^{6(g-1)}$$

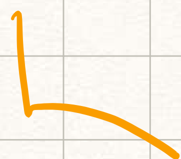
Fenchel Nielsen coord.

# Pythagoras Thm

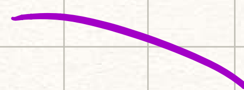
$$\cosh(c) = \cosh(a)\cosh(b)$$




PT: it costs the same to do



than



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