Undecidability in Groups

· Notes online: → @pmms / ~mcc56 / 2015-2016 / part III infinite / notes / part III FINAL.pdf

↑ numbering going to match PDF

§1. Amalgamated Products, HNN extensions & Turing Machine

Today: tech. tools

Thurs: Overview of construct. taking comp device & realizing it in fp group

Fri: Good results – Lots of things we can't compute

Def (5.1)

Let G, H grp, $A \leq G$, $B \leq H$, w/ $\varphi : A \xrightarrow{\cong} B$ isom.

the free amalgamation is

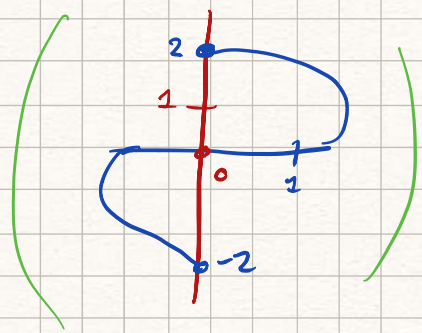
$$G *_{\varphi} H := \frac{G * H}{\langle\langle a = \varphi(a) \ \forall a \in A \rangle\rangle}$$

so can "push" element in prod if want to.

id A w/ B

"stick $A \leq G$ to $B \leq H$"

ex: $\mathbb{Z} * \mathbb{Z}$ w/ $2\mathbb{Z} \xrightarrow{\varphi} 2\mathbb{Z}$



Guess.

<u>Free prod</u> $\mathbb{Z} * \mathbb{Z} = F_2$

$$(\underset{x}{C_3} * \underset{y}{C_3}) *_{\varphi} \mathbb{Z}$$

$$\varphi : (xy) \longrightarrow (z^3)$$

<u>Q</u>: What do elements in $G *_\varphi H$ look like?

<u>Def (2.11)</u>

Let G, H grp. A P-reduced (product) sequence in $G * H$ is a finite seq of elements $g_1, \dots, g_n \in G \sqcup H$ st

1) No $g_i = e$ in G or H

2) No successive g_i, g_{i+1} are both in G (or both in H)

<u>Fact</u>: in free prod, every element has a unique P-reduced sequence.

Fact: in a A-reduced seq $c_1 \dots c_n \in G *_\varphi H$,
(amalgamated)

is T-reduced sequence st $\forall i > 1$,
we have that c_i is in A or B $(i > 1)$.

Idea: Can turn any T-red sequence in $G *_\varphi H$ into a A-red sequence by "absorbing" elements of A & B using φ w/o changing the elements it represents in $G *_\varphi H$.

ex: $g_1 h_1 \underbrace{a}_{\downarrow \varphi(a)} h_2 g_2$

$g_1 \underbrace{h_1 \varphi(a) h_2}_{H} g_2$

Thm (5.14) If $c_1 \dots c_n$ is A red in $G *_\varphi H$ and $n \geq 1$, then the element it represents is $\neq e$ in $G *_\varphi H$.

In particular, $G, H \hookrightarrow G *_\varphi H$ (take $n = 1$).

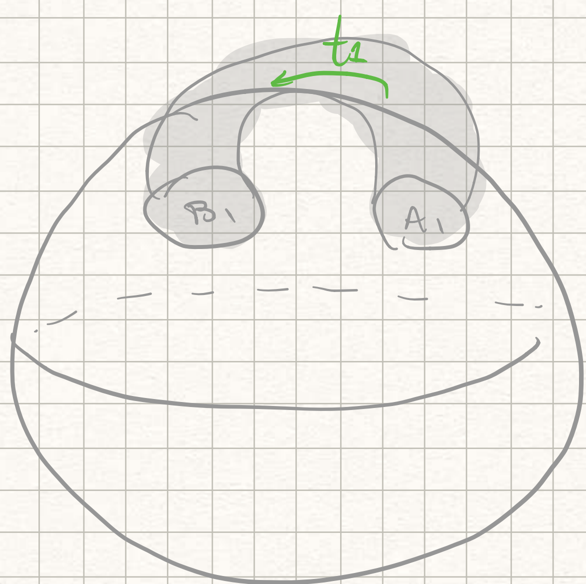
i.e., new way of embedding 2 subgrps into bigger group.

Def (5.15) Let G grp, A_i, B_i pairs of isom subgroup of G via $\varphi_i : A_i \xrightarrow{\cong} B_i$ $(1 \le i \le n)$.

We define the multiple HNN-extension $G *_{\varphi_1 \dots \varphi_n}$ w/ its base grp & stable letters $t_1, \dots, t_n$ ($\langle t_i \rangle$ copy of $\mathbb{Z}$):

$$G *_{\varphi_1 \dots \varphi_n} := \frac{G * \langle t_1 \rangle * \cdots * \langle t_n \rangle}{\langle\langle t_1 a t_1^{-1} = \varphi_1(a) \ \forall a \in A_1, \ \dots, \ t_n a t_n^{-1} = \varphi_n(a) \ \forall a \in A \rangle\rangle}$$

ie take free prod of G w/ F_n then id each A_i w/ its image $\varphi_i(A_i)$ $(= B_i)$ via conjugation by t_i.



Q: Why not stick A_i to B_i?

A: Would fold the whole grp.

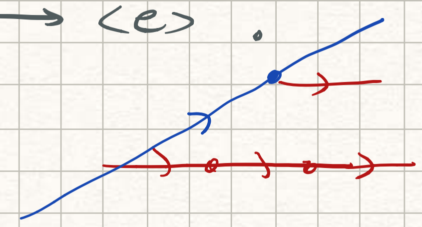
ex: $\mathbb{Z}$ $\qquad \langle a^2 \rangle = 2\mathbb{Z}$
a $\qquad \langle a^3 \rangle = 3\mathbb{Z}$

$$\varphi : (a^2) \longrightarrow (a^3)$$

$$a^2 \longmapsto a^3$$

if you stuck a^2 to a^3;

$$\langle a \mid a^2 = a^3 \rangle \longrightarrow \langle e \rangle .$$



Now,

$$\mathbb{Z} *_\varphi = \langle a, t \mid t a^2 t^{-1} = a^3 \rangle$$

Lem (5.16) An n-stage multi HNN extension is isom. to doing the successive single HNN-ext.

Def (5.17) Let $G *_{\varphi_1 \ldots \varphi_n}$ $(\varphi_i : A_i \to B_i)$ be HNN ext. w/ stable $t_1, \ldots, t_n$.

- An M-HNN seq. (m= multiple) is a seq of the form:

$$g_0 \, t_{j_1}^{\varepsilon_1} \, g_1 \cdots t_{j_m}^{\varepsilon_m} \, g_m$$

can be e if want t_i^3

where $m \geq 0$, each $g_i \in G$, $\varepsilon_i \in \{\pm 1\}$, $j_e \in \{1, \ldots, n\}$

- A ti-pinch of an MHNN seq is a subseq of the form $t_i g_i t_i^{-1}$ for $g \in A_i$ or $t_i^{-1} g_i t_i$ for $g_i \in B_i$.

- An MHNN seq. is reduced if it contains no t_i-pinch. for any $i \in [n]$.

Fact: Any elt of $G *_{\varphi_1 \dots \varphi_n}$ can be rep by a reduced HNN seq. (replace all pinches).

Thm (5.18) [Britton's lem]
If the MHNN seq $g_0 t_{j_1}^{\varepsilon_1} \cdots t_{j_m}^{\varepsilon_m} g_m$ contains NO pinches & $m \geq 1$, ~~then~~ it is NOT ~~the~~ id. in $G *_{\varphi_1 \dots \varphi_n}$. (nor it is even in G).

Thm (5.6 & 5.7) Each elmt of $G *_{\varphi_1 \dots \varphi_n}$ has a unique normal form (red. MH seq).
making use of coset rep.

Thus, G embeds in $G *_{\varphi_1 \dots \varphi_n}$ $(g \mapsto g)$.

Prop (5.2) If G fp, $A \le G$, $B \le H$, $C \le G$

$\varphi : A \to B$
$\psi : A \to C$

then $G *_\varphi H$ is fp

$G *_\psi$ is fp

Idea: using HNN ext, can embed Turing machine $\mathbb{Z} * (\mathbb{Z} \times \mathbb{Z})$.